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Theory for Predicting CPF Bow-Flare Impulsive Loads

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Contract number: W7707-021328/001/HAL

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Defence R&D Canada – Atlantic

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Canada

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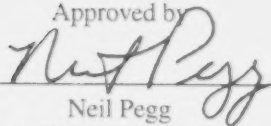
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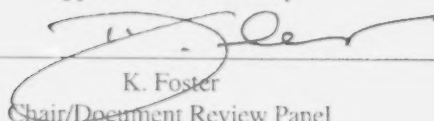
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Abstract

This report presents the theoretical background and the numerical scheme for predicting HALIFAX class bow-flare impulsive loads. The hull geometry of HALIFAX class is first approximated by a NURBS surface, from which any hull cross section can be determined for computation of two-dimensional slamming loads. Solution of the nonlinear free surface is a key element of the developed method. Validation of free surface computations with existing analytical solutions indicates good agreement.

Developed software includes a graphical user interface (GUI) and visualization using the Qt GUI library and OpenGL rendering library.

Résumé

Le présent rapport présente une explication théorique et un schéma numérique pour prédire les contraintes impulsives créées par l'effet de gifle sur le redan de la proue sur les navires de la classe HALIFAX. La géométrie de la coque des navires de cette classe est d'abord évaluée de façon approximative par une surface à représentation NURBS, d'où une coupe transversale de coque peut être déterminée par le calcul de deux contraintes de gifle bidimensionnelles. La solution d'une surface libre non linéaire est un élément clé de la méthode mise au point. La validation des calculs de surface libre avec les solutions analytiques existantes indiquent une bonne corrélation.

Le logiciel développé comprend une interface graphique et une visualisation au moyen de la bibliothèque de données Qt GUI et de la bibliothèque de présentation OpenGL.

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Executive summary

Introduction

Slamming forces must be considered for the safe design and operation of naval vessels. For frigate hull forms, bow-flare slamming can induce significant loads. This report describes the theory for predicting flow and pressures arising from bow-flare slamming. A separate report describes software developed using the theory.

Principal Results

The developed theory predicts flow and pressures from bow-flare slamming by assuming two-dimensional potential flow. Deformation of the free surface and development of jet flow are included in the solution. Comparisons with an analytical method for wedges entering water give excellent agreement.

Significance of Results

The developed method appears to be promising for predicting bow-flare slamming on two-dimensional ship sections. Validation with experimental data and examination of required computational times are recommended for the developed method.

Future Plans

Ongoing work is considering various methods for incorporating slamming into ship motion and sea load predictions.

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Sommaire

Introduction

Il faut tenir compte des forces d'effet de gifle pour concevoir et exploiter des navires de guerre en toute sécurité. Dans le cas des formes de coque des frégates, l'effet de gifle sur le redan de proue peut créer des contraintes importantes. Le présent rapport décrit une théorie permettant de prédire l'écoulement et les pressions résultant de cet effet de gifle. Un autre rapport décrit le logiciel mis au point qui fait appel à cette théorie.

Résultats principaux

La théorie élaborée prédit l'écoulement et les pressions créés par l'effet de gifle sur le redan de proue en supposant que l'écoulement potentiel est à deux dimensions. La déformation de la surface libre et la création d'un écoulement plein sont comprises dans la solution. Des comparaisons avec une méthode analytique sur des coins pénétrant dans l'eau donne une excellente corrélation.

Importance des résultats

La méthode élaborée semble prometteuse pour la prédiction des effets de gifle sur le redan de la proue pour des sections bidimensionnelles de navire. On recommande une validation avec des données expérimentales et un examen des durées de calcul nécessaires pour cette méthode.

Travaux ultérieurs prévus

Les travaux en cours envisagent diverses méthodes pour incorporer les effets de gifle sur les mouvements du navires et les prédictions sur les charges de mer.

J.M. Chuang and Wu Zhu; 2004; Theory for Predicting CPF Bow-Flare Impulsive Loads; DRDC Atlantic CR 2004-182; R & D pour la défense Canada – Atlantique.

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1 Introduction

In this contract, a unique software package that integrates hydrodynamic computation and interactive visualization is developed. The theoretical construct of this software package is derived from a computer aided ship/boat design package, namely, DalShip [1]. It makes use of NURBS (Non-Uniform Rational B-Spline), Qt (GUI-toolkits) and OpenGL (3-D rendering library) to develop and visualize a mathematical description of a 3-D ship hull. Consequently, the geometry of arbitrary cross section around bow used in the impulsive load computation can be obtained very easily and accurately. The main features of DalShip are as follows: firstly, the offsets of a given ship are used to find the fundamental control polygonal net to describe the global surface. With the help of a special arrangement for the knots and control points, a ship's surface can be generated very quickly. Secondly, the smoothness and continuity of the ship's surface are guaranteed from three specially designed GUI's where the ship's lines can be faired by moving the control points of NURBS surface. As an application of DalShip, the impulsive load is predicted at several cross sections near CPF's bow.

The objective of this contract is mainly to find quantified free surface, pressure distribution and slamming force on arbitrary cross section around the bow of CPF when it enters calm water. In order to simulate the CPF's bow re-entering the water in heavy seas, it is assumed that the velocity of the body is equal to the relative velocity of the body and the moving free surface. This assumption has been verified by Ridley [2]. The simplest water entry problem, i.e. the wedge entry into calm water, has been extensively studied by many researchers. The theoretical analysis of similarity flow of wedge entry was first conducted by Wagner [3]. Armand & Cointe [4], Cointe [5] and Howison, Ockendon & Wilson [6] extended Wagner's theory to analyze the wedge with small deadrise using matched asymptotic expansions. Based on Wagner's function, Dobrovol'skaya [7] developed an analytical solution in terms of a nonlinear integral equation for the problem of the uniform symmetrical entry of a wedge into calm water. Based on the work of Vinje and Brevig [8], Greenhow [9] presented numerical results for a wedge entry into calm water using Cauchy's formula and Euler-Lagrangian method. His numerical approach took both gravity and non-linear free surface condition into account. The agreement with the self-similarity assumed in earlier theories was good for wedge half-angles $\alpha < \frac{\pi}{6}$, but poor for larger wedge angles due to severe numerical difficulties. Zhao & Faltinsen [10] gave a numerical method for studying water entry of wedge, which was based on the traditional boundary element method with constant source panel. Due to the fact that the center of the constant element is used as the collocation point, the motion of the fluid particle at the intersection between the wedge surface and free surface cannot be traced in the Euler-Lagrangian method. Zhao & Faltinsen thus cut the jet tip at the intersection of free and body surfaces

with two small constant elements. Since at this intersection point, there are two boundary conditions to be satisfied, it causes difficulty in the numerical simulation. The same problem arises at the intersection between the wave maker and free surface in the wave maker problem. Lin, Newman and Yue [11] presented an approach to resolve this problem. It is that the boundary integral equation derived from Cauchy's formula is discretized with $N-1$ linear elements. Thus, there are N grid points which produce N algebraic equations to solve for N unknowns. Since the linear element is used, the two intersection points between body and free surface are collocation points. Due to the fact that both ϕ and ψ are given at these points, the terms associated with these points can be moved to the right-hand side of the equations, thus the number of algebraic equations reduces from N to $N-2$. It shows that this approach gives good numerical results for wavemaker problem. It can be extended to the water entry problem for handling the intersection points. However, all of the numerical methods mentioned above is based on the traditional boundary element method with local shape functions. The disadvantage is seen to be on the efficiency and accuracy issue due to the singular behavior of the kernel and the approximation of curved boundary with low-order polynomials.

In this contract, a very simple and accurate numerical scheme developed from desingularized Cauchy's formula by Chuang [12] is adopted to solve water entry problems. The special features of this numerical approach are that

1. The coefficient matrix of the discretized equations only relates the distances between source and field points, and; the normal vectors and weighting factors of Gaussian quadrature at source points. The computation of the coefficient matrix is thus reduced to minimum.
2. Since there is no singularity involved in the integral, the high accuracy of the numerical integration is guaranteed by a proper Gaussian quadrature [13].
3. Since the global property of boundary geometry is involved in the numerical computation, if the boundary has a mathematical description, the error of the numerical solution due to the approximation of the boundary geometry is almost negligible.
4. The global feature of the numerical scheme can be easily implemented to study the behavior and numerical treatment of corner singularity.

To validate our numerical scheme used for solving nonlinear boundary value problem of CPF bow-flare impulsive loads, a non-linear transient water waves in a rectangle container is first solved. The numerical results agreed with analytical solution very well.

In this report, Section 2 describes an approach to approximate a given ship hull with NURBS surface. The similarity transform of wedge entry problem from time-dependent to time independent boundary value problem is derived in Section 3. Section 4.1 and 4.2 introduce the desingularized Cauchy's formula for interior potential and half-space potential problems, respectively. The versatility of this desingularized Cauchy's formula is explained. The discretization technique using global Gaussian quadrature is also derived. It turns out that the discretized equations involve the boundary values of not only the real and imaginary parts of the analytic function but also their tangential derivatives. In Section 4.3, the numerical treatment of corner singularity is dealt with in detail. The behavior of the tangential derivatives of real and imaginary parts of the analytic function around the corner is first investigated. Collocating two points nearby corner is then proposed for the numerical implementation of Gaussian quadrature. Based on the double-node concept and a limiting process, a numerical scheme for using corner point as a collocation point is finally developed. Due to the fact that this research is dealt with time domain problem, Euler-Lagrangian method is introduced in Section 5. In order to find an accurate the pressure distribution on CPF's bow, the contribution to pressure distribution from the $\partial\phi/\partial t$ -term is studied in Section 6. Section 7 gives two numerical examples. The first one is for testing the time-march scheme of three numerical methods for the nonlinear transient wave in a rectangular container. The numerical results demonstrate the robustness of our numerical scheme. The second example is the wedge entry problem. Section 8 is reserved to present the numerical results of predicting CPF bow-flare impulsive loads. The last section, Section 9, will give conclusions and remarks.

2 Approximation of Ship Hull with NURBS Surface

2.1 Curve Fitting with NURBS

The NURBS curve is described as [14]

$$\vec{x}(\bar{u}) = \frac{\sum_{i=0}^n N_i^{(p)}(\bar{u}) w_i \vec{P}_i}{\sum_{i=0}^n N_i^{(p)}(\bar{u}) w_i} \quad (1)$$

where $\vec{x}$ denotes the position vector of a point (x, y, z) on a NURBS curve, $N_i^{(p)}$ and w_i are the p -degree basis function and weight, respectively, corresponding to the control point $\vec{P}_i$ located at (X_i, Y_i, Z_i) and $\bar{u}$ is the parameter which is between 0 and 1.

If a set of $(n+1)$ data points, $\vec{x}_k$, $k = 0, \dots, n$, on a curve is given, the $(n+1)$

control points $\vec{P}_i$ in Eqn. (1) are obviously unknowns to be found. In order to find $(n+1)$ linear equations to solve for $(n+1)$ unknowns, the parameter $\bar{u}_k$ in Eqn. (1) corresponding to each given data point $\vec{r}_k$ has to be assigned. Since $0 \leq \bar{u} \leq 1$, the normalized chord length is widely used for determining $\bar{u}_k$:

$$\bar{u}_0 = 0, \quad \bar{u}_n = 1, \quad \text{and} \quad \bar{u}_k = \bar{u}_{k-1} + \frac{|\vec{x}_k - \vec{x}_{k-1}|}{d}, \quad k = 1, \dots, n-1 \quad (2)$$

where

$$d = \sum_{k=1}^n |\vec{x}_k - \vec{x}_{k-1}|$$

After $\bar{u}_k$ is defined for each given data point $\vec{x}_k$, the $(n+p+2)$ components u_i 's, for $i = 0, 1, 2, \dots, m'$, of the nonuniform knot vector U shown in Eqn. (3) can be computed from Eqn. (4).

$$U = \{u_0, \dots, u_p, u_{p+1}, \dots, u_{m'-p-1}, u_{m'-p}, \dots, u_{m'}\} \quad (3)$$

$$u_0 = \dots = u_p = 0 \quad \text{and} \quad u_{m'-p} = \dots = u_{m'} = 1$$

$$u_{j+p} = \frac{1}{p} \sum_{i=j}^{j+p-1} \bar{u}_i, \quad j = 1, \dots, n-p \quad (4)$$

where $m' = n+p+1$. Notice that the nonuniform knot vector U is used to compute the basis function $N_i^{(p)}(\bar{u}_k)$.

Thus, based on Eqn. (1) with unit weight for each control point $\vec{P}_i$, a system of $n+1$ linear equations can be formed:

$$\begin{bmatrix} N_0^{(p)}(\bar{u}_0) & N_1^{(p)}(\bar{u}_0) & N_2^{(p)}(\bar{u}_0) & \dots & N_n^{(p)}(\bar{u}_0) \\ N_0^{(p)}(\bar{u}_1) & N_1^{(p)}(\bar{u}_1) & N_2^{(p)}(\bar{u}_1) & \dots & N_n^{(p)}(\bar{u}_1) \\ N_0^{(p)}(\bar{u}_2) & N_1^{(p)}(\bar{u}_2) & N_2^{(p)}(\bar{u}_2) & \dots & N_n^{(p)}(\bar{u}_2) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ N_0^{(p)}(\bar{u}_n) & N_1^{(p)}(\bar{u}_n) & N_2^{(p)}(\bar{u}_n) & \dots & N_n^{(p)}(\bar{u}_n) \end{bmatrix} \cdot \begin{bmatrix} \vec{P}_0 \\ \vec{P}_1 \\ \vec{P}_2 \\ \vdots \\ \vdots \\ \vec{P}_n \end{bmatrix} = \begin{bmatrix} \vec{x}_0 \\ \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vdots \\ \vec{x}_n \end{bmatrix} \quad (5)$$

For each individual ship's cross section, the degree $p = 3$ of NURBS basis functions is used to generate the control polygon.

2.2 Surface Fitting with NURBS

The description of a NURBS surface is given by

$$\vec{x}(\bar{u}, \bar{v}) = \frac{\sum_{i=0}^n \sum_{j=0}^m N_i^{(p)}(\bar{u}) N_j^{(q)}(\bar{v}) w_{i,j} \vec{P}_{i,j}}{\sum_{i=0}^n \sum_{j=0}^m N_i^{(p)}(\bar{u}) N_j^{(q)}(\bar{v}) w_{i,j}} \quad (6)$$

where $\vec{x}(\bar{u}, \bar{v})$ is the position vector for the point (x, y, z) on the NURBS surface, $N_i^{(p)}(\bar{u})$ and $N_j^{(q)}(\bar{v})$ are p - and q -degree basis functions corresponding to the control point $\vec{P}_{i,j}$ located at $(X_{i,j}, Y_{i,j}, Z_{i,j})$ associated with weight $w_{i,j}$; and, $\bar{u}$ and $\bar{v}$ are parameters, $0 \leq \bar{u} \leq 1$ and $0 \leq \bar{v} \leq 1$.

If a set of $(n+1) \times (m+1)$ data points, $\vec{x}_{k,l}$, for $k = 0, \dots, n$ and $l = 0, \dots, m$, on a curved surface is given, then $(n+1) \times (m+1)$ control points $\vec{P}_{i,j}$ in Eqn. (6) are to be found. Similar to the NURBS curve fitting, a pair of parameters $(\bar{u}_k, \bar{v}_l)$ associated with each given data point must be defined and the knot vectors U and V formed for computing the basis functions. Based on Eqn. (2), the parameters, $\bar{u}_0^l, \dots, \bar{u}_n^l$ for each l are computed first, then $\bar{u}_k$ is obtained by averaging all of $\bar{u}_k^l, l = 0, \dots, m$:

$$\bar{u}_k = \frac{1}{(m+1)} \sum_{l=0}^m \bar{u}_k^l, \quad k = 0, \dots, n \quad (7)$$

The same procedure is repeated for calculating $\bar{v}_l, l = 0, \dots, m$. Once parameters $(\bar{u}_k, \bar{v}_l)$ are computed, the nonuniform knot vectors U and V can be formed from Eqn. (4). Due to the fact that the NURBS surface, $\vec{x}(\bar{u}, \bar{v})$, is a tensor-product surface, the system of linear equations developed from Eqn. (6) with unit weight $w_{i,j}$ for solving control points $\vec{P}_{i,j}$ can be split into two subsystems as shown in [14] for detail.

2.3 Special Considerations of NURBS Surface Fitting for Ship's Geometry

Now that control points $\vec{P}_{i,j}$ associated with the pair of parameters $(\bar{u}_k, \bar{v}_l)$ for each given point $\vec{x}_{k,l}$ and knot vectors, U and V , for generating a NURBS surface to approximate the ship's hull are determined (see previous section), the fine tuning of the obtained NURBS surface begins. Reason: special requirements of the new ship's geometry, like the shape of stem, midship and stern. The detailed approach can be found in [1].

2.4 Further Interactive Refinement of Ship Lines

In addition to the considerations of NURBS surface fitting for certain features of ship's geometry, an interactive approach has also been developed to modify the NURBS surface. This is based on the control points of NURBS surface, OpenGL [15] which is a 3D rendering graphics library and the Qt [16], a GUI (Graphic User Interface) development toolkits. In order to carry out a small change of ship lines which have been approximated by a NURBS surface, the user clicks a control

point with a mouse and drags. The control point will move, and the weight just corresponding to the control point can be changed. A small local change of the ship geometry within the control polygon will be thus carried out simultaneously. This type of real-time modification can be carried out in three different plans, namely body plan, sheer plan and water plan. The NURBS representation of CPF's surface is illustrated in Fig. 1.

2.5 NURBS Surface Interpolation

Based on the mathematical description of the NURBS surface as shown in Eqn. (6), the point (x, y, z) on the surface of the new ship can be expressed in terms of three equations:

$$x(\bar{u}, \bar{v}) = \frac{\bar{N}_x(\bar{u}, \bar{v})}{\bar{D}(\bar{u}, \bar{v})}, \quad y(\bar{u}, \bar{v}) = \frac{\bar{N}_y(\bar{u}, \bar{v})}{\bar{D}(\bar{u}, \bar{v})}, \quad z(\bar{u}, \bar{v}) = \frac{\bar{N}_z(\bar{u}, \bar{v})}{\bar{D}(\bar{u}, \bar{v})} \quad (8)$$

where

$$\begin{aligned} \bar{N}_x(\bar{u}, \bar{v}) &= \sum_{i=0}^n \sum_{j=0}^m N_i^{(p)}(\bar{u}) N_j^{(q)}(\bar{v}) w_{i,j} X_{i,j} \\ \bar{N}_y(\bar{u}, \bar{v}) &= \sum_{i=0}^n \sum_{j=0}^m N_i^{(p)}(\bar{u}) N_j^{(q)}(\bar{v}) w_{i,j} Y_{i,j} \\ \bar{N}_z(\bar{u}, \bar{v}) &= \sum_{i=0}^n \sum_{j=0}^m N_i^{(p)}(\bar{u}) N_j^{(q)}(\bar{v}) w_{i,j} Z_{i,j} \\ \bar{D}(\bar{u}, \bar{v}) &= \sum_{i=0}^n \sum_{j=0}^m N_i^{(p)}(\bar{u}) N_j^{(q)}(\bar{v}) w_{i,j} \end{aligned}$$

Notice that $X_{i,j}$, $Y_{i,j}$ and $Z_{i,j}$ are coordinates of the control point $\vec{P}_{i,j}$.

Obviously, if two coordinates of a point on the new ship's surface are given, the third coordinate can be found. For example, the coordinate x (in ship's length direction) and z (in ship's depth direction) are given for a specific point on the surface. Then the parametric variables $\bar{u}$ and $\bar{v}$ of the NURBS surface can be found from two nonlinear equations in Eqn. (8):

$$F(\bar{u}, \bar{v}) = \frac{\bar{N}_x(\bar{u}, \bar{v})}{\bar{D}(\bar{u}, \bar{v})} - x = 0, \quad G(\bar{u}, \bar{v}) = \frac{\bar{N}_z(\bar{u}, \bar{v})}{\bar{D}(\bar{u}, \bar{v})} - z = 0 \quad (9)$$

After $\bar{u}$ and $\bar{v}$ are found, the coordinate y of this point can be computed from the second equation in Eqn. (8). The set of nonlinear equations, Eqn. (9) can be solved by Newton-Raphson's iteration. A similar approach can be applied to the case if x and y , or y and z coordinates are given for a point x on the hull surface.

2.6 CPF Geometry

With NURBS surface fitting, CPF's body surface in solid is shown in Fig. 1 while its mesh expression is depicted in Fig. 2. Fig. 3 shows global surface control polygon net of CPF's surface.

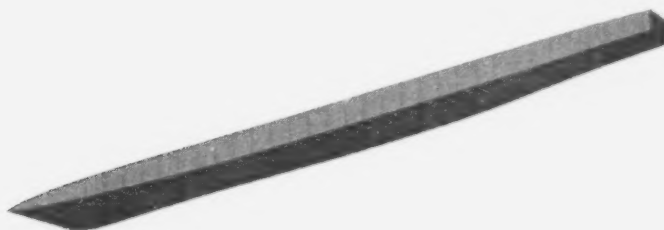


Figure 1: Solid

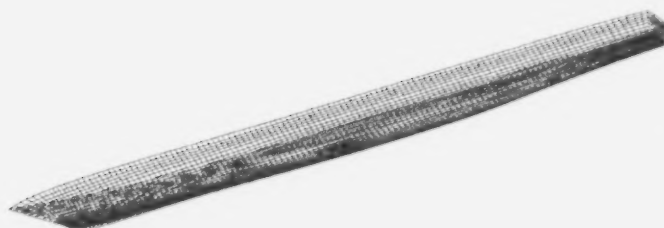


Figure 2: Frame



Figure 3: control point

3 Similarity Transformation of Wedge Entry Problem

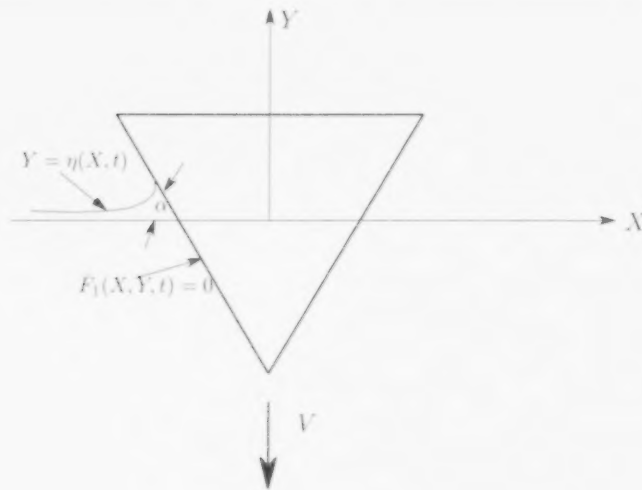


Figure 4: Water Entry of a Wedge

The mathematical formulation of the water entry of a wedge with an deadrise angle α falling with a constant velocity shown in Fig. 4 is as follows:

1. Body Surface Description

$$\bar{F}_1(X, Y, t) = Y + Vt + X \tan \alpha = 0 \quad (10)$$

2. Free Surface Description

$$Y = \eta(X, t) \quad \text{or} \quad \bar{F}_2(X, Y, t) = Y - \eta(X, t) = 0 \quad (11)$$

3. Governing Equation

$$\nabla^2 \Phi(X, Y, t) = 0 \quad (12)$$

4. Body Surface Condition

$$\frac{\partial \Phi}{\partial n} = -V n_y = V \frac{\partial X}{\partial s} \quad \text{on} \quad \bar{F}_1(X, Y, t) = 0 \quad (13)$$

5. Dynamic Free Surface Condition (Without Gravitational Force)

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla \Phi = 0 \quad \text{on} \quad \bar{F}_2(X, Y, t) = 0 \quad (14)$$

6. Kinematic Free Surface Condition

$$\frac{\partial \bar{F}_2}{\partial t} + \nabla \Phi \cdot \nabla \bar{F}_2 = 0 \quad \text{or} \quad \frac{\partial \Phi}{\partial n} = -\frac{1}{|\nabla \bar{F}_2|} \frac{\partial \bar{F}_2}{\partial t} \quad \text{on} \quad \bar{F}_2(X, Y, t) = 0 \quad (15)$$

Notice that

$$\nabla \bar{F}_2 = -\bar{\eta}_x \vec{i} + \vec{j}, \quad |\nabla \bar{F}_2| = \sqrt{1 + \bar{\eta}_x^2}, \quad \text{and} \quad \frac{\partial \bar{F}_2}{\partial t} = -\bar{\eta}_t \quad (16)$$

We now define a set of non-dimensional parameter such that

$$x = \frac{X}{Vt}, \quad y = \frac{Y}{Vt}, \quad \phi(x, y) = \frac{\Phi(X, Y, t)}{V^2 t}, \quad \text{and} \quad \eta(x, y) = \frac{\bar{\eta}(X, Y, t)}{Vt} \quad (17)$$

The transformed problem is then time independent as shown below:

1. Body Surface Description

$$F_1(x, y) = y + 1 + x \tan \alpha = 0 \quad (18)$$

Notice that

$$\bar{F}_1(X, Y, t) = Vty + Vt + Vtx \tan \alpha = Vt(y + 1 + x \tan \alpha) = VtF_1(x, y) = 0 \quad (19)$$

2. Free Surface Description

$$y = \eta(x) \quad \text{or} \quad F_2(x, y) = y - \eta(x) = 0 \quad (20)$$

3. Governing Equation

$$\nabla^2 \phi(x, y) = 0 \quad (21)$$

4. Body Surface Condition

$$\frac{\partial \phi}{\partial n} = -n_y = \frac{\partial \phi}{\partial s} \quad \text{on} \quad F_1(x, y) = 0 \quad (22)$$

5. Dynamic Free Surface Condition

$$\phi + \frac{1}{2} \nabla \phi \cdot \nabla \phi - x \frac{\partial \phi}{\partial x} - \eta(x) \frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad F_2(x, y) = 0 \quad (23)$$

6. Kinematic Free Surface Condition

$$\frac{\partial \phi}{\partial y} - \frac{d\eta}{dx} \frac{\partial \phi}{\partial x} + x \frac{d\eta}{dx} - \eta(x) = 0 \quad (24)$$

or

$$\frac{\partial \phi}{\partial n} = -\frac{1}{\sqrt{1+\eta_x^2}} \left\{ x\eta_x - \eta(x) \right\} \quad (25)$$

Notice that

1. Since

$$\bar{\eta}(X, t) = Vt\eta(x), \quad \frac{\partial \bar{\eta}}{\partial t} = V\eta(x) + Vt \frac{\partial \eta}{\partial x} \frac{\partial x}{\partial t} \bigg|_{X=Vtx} \quad (26)$$

and

$$x = \frac{X}{Vt}, \quad \frac{\partial x}{\partial t} = -\frac{X}{Vt^2} \bigg|_{X=Vtx} = -\frac{Vtx}{Vt^2} = -\frac{x}{t} \quad (27)$$

Thus, we obtain

$$\frac{\partial \bar{\eta}}{\partial t} = V \left\{ \eta(x) - \frac{\partial \eta}{\partial x} x \right\} \quad (28)$$

2. Similarly, due to the fact that

$$\begin{aligned} \Phi(X, Y, t) &= V^2 t \phi(x, y), \\ \frac{\partial \Phi}{\partial t} &= V^2 \phi(x) + V^2 t \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial t} \bigg|_{X=Vtx} + V^2 t \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial t} \bigg|_{Y=Vty} \end{aligned} \quad (29)$$

and

$$\begin{aligned} x &= \frac{X}{Vt}, \\ y &= \frac{Y}{Vt}, \\ \frac{\partial x}{\partial t} &= -\frac{X}{Vt^2} \bigg|_{X=Vtx} \\ &= -\frac{Vtx}{Vt^2} = -\frac{x}{t}, \end{aligned}$$

$$\begin{aligned}
 \frac{\partial y}{\partial t} &= -\frac{Y}{Vt^2} \bigg|_{Y=Vty} \\
 &= -\frac{Vty}{Vt^2} \\
 &= -\frac{y}{t}
 \end{aligned}
 \tag{30}$$

It follows that

$$\frac{\partial \Phi}{\partial t} = V^2 \left\{ \phi - \frac{\partial \phi}{\partial x} x - \frac{\partial \phi}{\partial y} y \right\}
 \tag{31}$$

On the free surface, $y = \eta(x)$, it follows that

$$\frac{\partial \Phi}{\partial t} = V^2 \left\{ \phi - \frac{\partial \phi}{\partial x} x - \frac{\partial \phi}{\partial y} \eta(x) \right\}
 \tag{32}$$

Based on this transformed boundary value problem, Dobrovol'skaya developed an analytical solution. Some investigations of the singular integral involved in the analytical solution are included in Appendix A.

4 Desingularized Cauchy's Formula

4.1 Interior Potential Problem



Figure 5: Domain R , Boundary S and Interior Angle α

Let us consider the Cauchy's formula for an analytic function $f(z)$ inside the domain R surrounded by a boundary curve S as shown in Fig. 5. Introducing the concept of the principal value of a Cauchy's integral [15] for $z \in S$, the formula can be written as:

$$\frac{c}{2\pi} f(z) = \frac{1}{2\pi i} \int_S \frac{f(z_0)}{z_0 - z} dz_0 \quad \text{for } z_0 \in S
 \tag{33}$$

where

$$c = \begin{cases} 2\pi & \text{for } z \in R \\ i\alpha & \text{for } z \in S \end{cases}$$

Notice that α is the interior angle between two tangents to the contour S at the point z (see also Fig. 5). Using the 'subtracting and adding back' technique, the Cauchy's formula can be written as

$$\frac{c}{2\pi} f(z) = \frac{1}{2\pi i} \int_S \frac{f(z_0) - f(z)}{z_0 - z} dz_0 + \frac{f(z)}{2\pi i} \int_S \frac{1}{z_0 - z} dz_0 \quad (34)$$

Due to the fact that

$$\int_S \frac{1}{z_0 - z} dz_0 = c i \quad \text{for } z_0 \in S \quad (35)$$

where c is the same as that shown in Eqn. (33), Eqn. (34) reduces to

$$0 = \frac{1}{2\pi i} \int_S \frac{f(z_0) - f(z)}{z_0 - z} dz_0 \quad \text{for } z_0 \in S \quad (36)$$

It is emphasized that the versatility of this equation is that it is valid at any point inside the domain R , on the smooth boundary and even at the corner point of the boundary.

On the boundary S , we assume that the function $f(z)$ satisfies the Hölder's condition so that the integrand in Eqn. (36) has a finite value when z_0 approaches z . Here, we may write

$$\lim_{z_0 \rightarrow z} \frac{f(z_0) - f(z)}{z_0 - z} = \left. \frac{df(z_0)}{dz_0} \right|_{z_0=z} = f'(z) \quad (37)$$

Since there is no singularity involved in Eqn. (36), the Gaussian quadrature with N Gaussian points, $z_j, j = 1, \dots, N$, can be applied directly to carry out the integral, which gives

$$\sum_{j=1}^N \frac{f_j - f(z)}{z_j - z} dz_j w_j = 0 \quad \text{for } z \neq z_j \quad (38)$$

where f_j and w_j are the function value of $f(z)$ and weighting factor, respectively, at Gaussian point z_j .

It is clear that if f_j 's are known at Gaussian points on the boundary, the function $f(z)$ can be easily and accurately computed at any point inside the domain R and on the boundary S from

$$f(z) = \frac{\sum_{j=1}^N f_j dz_j w_j / (z_j - z)}{\sum_{j=1}^N dz_j w_j / (z_j - z)} \quad \text{for } z \neq z_j \quad (39)$$

For the boundary value problem in the potential theory, generally speaking, either the real or the imaginary part of f_j is unknown. So that before Eqn. (39) can be applied, the unknowns at Gaussian points have to be found.

4.1.1 Discretization of Integral Equation Using Gaussian Quadrature

To solve the unknown part of $f(z)$ on the boundary, we let the field point z be located on the boundary S .

Expressing the integral of Eqn. (36) in terms of the arc length of S gives

$$0 = \frac{1}{2\pi i} \int_S \left\{ \frac{f(z_0(s)) - f(z(s))}{z_0(s) - z(s)} \right\} \frac{dz_0}{ds} ds \quad \text{for } z, z_0 \in S \quad (40)$$

Accordingly, the limit when z_0 approaches z on the boundary is found to be

$$\lim_{z_0 \rightarrow z} \left\{ \frac{f(z_0(s)) - f(z(s))}{z_0(s) - z(s)} \right\} \frac{dz_0}{ds} = \frac{df}{dz_0} \frac{dz_0}{ds} = \frac{df}{ds} \quad (41)$$

Applying Gaussian quadrature with N Gaussian points which are also selected as the collocation points of the integral equation to carry out the integral of Eqn. (40) leads to a system of complex equations:

$$\sum_{j=1, j \neq i}^N \left\{ \frac{f_j - f_i}{z_j - z_i} \right\} \frac{dz_0}{ds} \Big|_j w_j + \frac{df}{ds} \Big|_i w_i = 0 \quad \text{for } i = 1, \dots, N \quad (42)$$

Notice that

$$\left\{ \frac{f_j - f_i}{z_j - z_i} \right\} \frac{dz_0}{ds} \Big|_j = \frac{df}{ds} \Big|_i \quad \text{when } j = i \quad (43)$$

We now let

$$\begin{aligned} f(z_0(s)) &= \phi(s) + i\psi(s), \quad \text{then} \quad \frac{df}{ds} = \frac{d\phi}{ds} + i \frac{d\psi}{ds} = \phi' + i\psi' \\ z_0(s) &= x_0(s) + iy_0(s), \quad \text{then} \quad \frac{dz_0}{ds} = \frac{dx_0}{ds} + i \frac{dy_0}{ds} = x'_0 + iy'_0 \end{aligned} \quad (44)$$

It is worthwhile to mention that the real and imaginary parts of dz_0/ds represent the components of the tangent vector $\vec{s}$ along the boundary curve S . Substituting

Eqn. (44) into Eqn. (42) gives

$$\sum_{j=1, j \neq i}^N \left\{ \frac{(\phi_j - \phi_i) + i(\psi_j - \psi_i)}{(x_j - x_i) + i(y_j - y_i)} \right\} (x'_j + iy'_j)w_j + (\phi'_i + i\psi'_i)w_i = 0 \text{ for } i = 1, \dots, N \quad (45)$$

Separating the real and imaginary parts, two sets of real equations are then obtained,

$$\begin{aligned} & \sum_{j=1, j \neq i}^N (H_{ij}y'_j - G_{ij}x'_j)(\psi_j - \psi_i)w_j \\ &= \sum_{j=1, j \neq i}^N (H_{ij}x'_j + G_{ij}y'_j)(\phi_j - \phi_i)w_j + w_i\phi'_i \end{aligned} \quad (46)$$

$$\begin{aligned} & \sum_{j=1, j \neq i}^N (H_{ij}y'_j - G_{ij}x'_j)(\phi_j - \phi_i)w_j \\ &= - \sum_{j=1, j \neq i}^N (H_{ij}x'_j + G_{ij}y'_j)(\psi_j - \psi_i)w_j - w_i\psi'_i \end{aligned} \quad (47)$$

where

$$H_{ij} = \frac{x_j - x_i}{(x_j - x_i)^2 + (y_j - y_i)^2}, \quad G_{ij} = \frac{y_j - y_i}{(x_j - x_i)^2 + (y_j - y_i)^2} \quad (48)$$

Eqn. (46) and Eqn. (47) can be also written in the following vector forms:

$$\sum_{j=1, j \neq i}^N (\psi_j - \psi_i) \frac{\vec{r}_{ij} \cdot \vec{n}_j}{r_{ij}^2} w_j = \sum_{j=1, j \neq i}^N (\phi_j - \phi_i) \frac{\vec{r}_{ij} \cdot \vec{s}_j}{r_{ij}^2} w_j + w_i\phi'_i \quad (49)$$

for $i = 1, \dots, N$

$$\sum_{j=1, j \neq i}^N (\phi_j - \phi_i) \frac{\vec{r}_{ij} \cdot \vec{n}_j}{r_{ij}^2} w_j = - \sum_{j=1, j \neq i}^N (\psi_j - \psi_i) \frac{\vec{r}_{ij} \cdot \vec{s}_j}{r_{ij}^2} w_j - w_i\psi'_i \quad (50)$$

for $i = 1, \dots, N$

where

$$\vec{r}_{ij} = (x_j - x_i)\vec{i} + (y_j - y_i)\vec{j}, \quad \vec{n}_j = y'_j\vec{i} - x'_j\vec{j} \quad \text{and} \quad \vec{s}_j = x'_j\vec{i} + y'_j\vec{j}$$

Casting Eqn. (46) and Eqn. (47) in matrix forms yields

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & \dots & A_{NN} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1N} \\ B_{21} & B_{22} & \dots & B_{2N} \\ \dots & \dots & \dots & \dots \\ B_{N1} & B_{N2} & \dots & B_{NN} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}$$

$$\begin{aligned}
& + \begin{pmatrix} w_1 \phi'_1 \\ w_2 \phi'_2 \\ \vdots \\ w_N \phi'_N \end{pmatrix} \quad (51) \\
& \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & \dots & A_{NN} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} = - \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1N} \\ B_{21} & B_{22} & \dots & B_{2N} \\ \dots & \dots & \dots & \dots \\ B_{N1} & B_{N2} & \dots & B_{NN} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \\
& - \begin{pmatrix} w_1 \psi'_1 \\ w_2 \psi'_2 \\ \vdots \\ w_N \psi'_N \end{pmatrix} \quad (52)
\end{aligned}$$

where

$$\begin{aligned}
A_{ij} &= \begin{cases} \frac{\tilde{r}_{ij} \tilde{r}_{ij}}{r_{ij}^2} w_j = (H_{ij} y_j' - G_{ij} x_j') w_j & j \neq i \\ -\sum_{j=1, j \neq i}^N \frac{\tilde{r}_{ij} \tilde{r}_{ij}}{r_{ij}^2} w_j = -\sum_{j=1, j \neq i}^N (H_{ij} y_j' - G_{ij} x_j') w_j & j = i \end{cases} \\
B_{ij} &= \begin{cases} \frac{\tilde{r}_{ij} \tilde{s}_{ij}}{r_{ij}^2} w_j = (H_{ij} x_j' + G_{ij} y_j') w_j & j \neq i \\ -\sum_{j=1, j \neq i}^N \frac{\tilde{r}_{ij} \tilde{s}_{ij}}{r_{ij}^2} w_j = -\sum_{j=1, j \neq i}^N (H_{ij} x_j' + G_{ij} y_j') w_j & j = i \end{cases}
\end{aligned}$$

From a computational point view, it is advantageous to compute A_{ij} and B_{ij} for $j \neq i$ first. The coefficients A_{ii} and B_{ii} for $i = 1, \dots, N$ can then be simply obtained from

$$A_{ii} = - \sum_{j=1, j \neq i}^N A_{ij}, \quad B_{ii} = - \sum_{j=1, j \neq i}^N B_{ij} \quad (53)$$

4.1.2 Application to Mixed Boundary Value Problem

If $\phi(Q) = h(Q)$ is described on part of the boundary, say S_1 , and $\partial\phi/\partial n(Q) = g(Q)$ is given on the other part, say S_2 , as shown in Fig. 6, then ϕ , ϕ' and ψ , ψ' are known on S_1 and S_2 , respectively. The boundary integral equations have to be split into two parts:

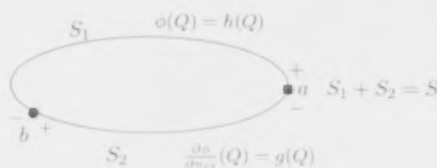


Figure 6: Mixed Boundary Value Problem

$$\begin{aligned} & \int_{S_1} + \int_{S_2} \{ \psi(Q) - \psi(P) \} \nabla_Q \left\{ \log \frac{1}{r_{PQ}} \right\} \cdot \vec{n}_Q dS_Q \\ &= \int_{S_1} + \int_{S_2} \{ \phi(Q) - \phi(P) \} \nabla_Q \left\{ \log \frac{1}{r_{PQ}} \right\} \cdot \vec{s}_Q dS_Q \\ & \int_{S_1} + \int_{S_2} \{ \phi(Q) - \phi(P) \} \nabla_Q \left\{ \log \frac{1}{r_{PQ}} \right\} \cdot \vec{n}_Q dS_Q \\ &= - \int_{S_1} - \int_{S_2} \{ \psi(Q) - \psi(P) \} \nabla_Q \left\{ \log \frac{1}{r_{PQ}} \right\} \cdot \vec{s}_Q dS_Q \end{aligned}$$

Applying Gaussian quadrature with N collocation points to the integral over S_1 and S_2 , separately, a similar set of equations as shown in Eqn. (51) and Eqn. (52) is obtained. After some matrix manipulations, i. e. moving all of the unknowns to the left hand side of equations, it follows that

$$\begin{pmatrix} A_{1,1} & \dots & A_{1,N} & -B_{1,N+1} & \dots & -B_{1,2N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{N,1} & \dots & A_{N,N} & -B_{N,N+1} & \dots & -B_{N,2N} \\ B_{N+1,1} & \dots & B_{N+1,N} & A_{N+1,N+1} & \dots & A_{N+1,2N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ B_{2N,1} & \dots & B_{2N,N} & A_{2N,N+1} & \dots & A_{2N,2N} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \\ \phi_{N+1} \\ \vdots \\ \phi_{2N} \end{pmatrix} = \begin{pmatrix} B_{1,1} & \dots & B_{1,N} & -A_{1,N+1} & \dots & -A_{1,2N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ B_{N,1} & \dots & B_{N,N} & -A_{N,N+1} & \dots & -A_{N,2N} \\ -A_{N+1,1} & \dots & -A_{N+1,N} & -B_{N+1,N+1} & \dots & -B_{N+1,2N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -A_{2N,1} & \dots & -A_{2N,N} & -B_{2N,N+1} & \dots & -B_{2N,2N} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_N \\ \psi_{N+1} \\ \vdots \\ \psi_{2N} \end{pmatrix}$$

$$+ \begin{pmatrix} w_1 h'_1 \\ \vdots \\ w_N h'_N \\ -w_1 g_{N+1} \\ \vdots \\ -w_N g_{2N} \end{pmatrix} \quad (54)$$

After ψ 's on S_1 and ϕ 's on S_2 are solved, ψ' on S_1 and ϕ' on S_2 can be obtained from

$$\begin{pmatrix} -w_1 \psi'_1 \\ \vdots \\ -w_N \psi'_N \\ w_1 \phi'_{N+1} \\ \vdots \\ w_N \phi'_{2N} \end{pmatrix} = \begin{pmatrix} A_{1,1} & \cdots & A_{1,N} & B_{1,N+1} & \cdots & B_{1,2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{N,1} & \cdots & A_{N,N} & B_{N,N+1} & \cdots & B_{N,2N} \\ -B_{N+1,1} & \cdots & -B_{N+1,N} & A_{N+1,N+1} & \cdots & A_{N+1,2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -B_{2N,1} & \cdots & -B_{2N,N} & A_{2N,N+1} & \cdots & A_{2N,2N} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_N \\ \psi_{N+1} \\ \vdots \\ \psi_{2N} \end{pmatrix} - \begin{pmatrix} -B_{1,1} & \cdots & -B_{1,N} & -A_{1,N+1} & \cdots & -A_{1,2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -B_{N,1} & \cdots & -B_{N,N} & -A_{N,N+1} & \cdots & -A_{N,2N} \\ -A_{N+1,1} & \cdots & -A_{N+1,N} & B_{N+1,N+1} & \cdots & B_{N+1,2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -A_{2N,1} & \cdots & -A_{2N,N} & B_{2N,N+1} & \cdots & B_{2N,2N} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \\ \phi_{N+1} \\ \vdots \\ \phi_{2N} \end{pmatrix} \quad (55)$$

4.2 Potential Problem for Half-Space

Similarly, for half-space problem, the desingularized Cauchy's formula can be written as

$$i\pi f(z) = - \int_{-\infty}^{+\infty} \frac{f(z_0) - f(z)}{z_0 - z} \frac{dz_0}{ds} ds \quad (56)$$

Accordingly the limit when z_0 approaches z on the boundary S is found to be

$$\lim_{z_0 \rightarrow z} \frac{f(z_0) - f(z)}{z_0 - z} \frac{dz_0}{ds} = \frac{df}{dz_0} \frac{dz_0}{ds} = \frac{df}{ds} \quad (57)$$

where s is the variable of arc length of boundary S .

The complex potential $f(z) = \phi(z) + i\psi(z)$, $z = x + iy$, is applicable for describing the fluid motion. Both the velocity potential ϕ and the stream function ψ satisfy Laplace's equation so that $f(z)$ is analytical in the fluid domain, therefore the Cauchy's formula is valid on this problem. $f(z)$ and z can be expressed in term of s . We let $f(z_0(s)) = \phi(s) + i\psi(s)$, $z_0(s) = x_0(s) + iy_0(s)$ then

$$\frac{df}{ds} = \frac{d\phi}{ds} + i \frac{d\psi}{ds} = \phi' + i\psi' \quad (58)$$

$$\frac{dz_0}{ds} = \frac{dx_0}{ds} + i \frac{dy_0}{ds} = x'_0 + iy'_0 \quad (59)$$

Separating Eqn. (56) into real and imaginary parts yield two integral equations:

$$-\pi\psi = \text{Re} \left\{ - \int_{-\infty}^{+\infty} \frac{f(z_0) - f(z)}{z_0 - z} \frac{dz_0}{ds} ds \right\} \quad (60)$$

$$\pi\phi = \text{Im} \left\{ - \int_{-\infty}^{+\infty} \frac{f(z_0) - f(z)}{z_0 - z} \frac{dz_0}{ds} ds \right\} \quad (61)$$

To solve the integral equations for ϕ and ψ , the Gaussian quadrature with N Gaussian points is applied to carry out the integral in Eqn. (60) and (61). It follows that

$$\pi\psi_i + \sum_{j=1, j \neq i}^N (\psi_j - \psi_i) \frac{\vec{r}_{ij} \cdot \vec{n}_j}{r_{ij}^2} w_j = \sum_{j=1, j \neq i}^N (\phi_j - \phi_i) \frac{\vec{r}_{ij} \cdot \vec{s}_j}{r_{ij}^2} w_j + w_i \phi'_i \quad (62)$$

for $i = 1, \dots, N$

$$\pi\phi_i + \sum_{j=1, j \neq i}^N (\phi_j - \phi_i) \frac{\vec{r}_{ij} \cdot \vec{n}_j}{r_{ij}^2} w_j = - \sum_{j=1, j \neq i}^N (\psi_j - \psi_i) \frac{\vec{r}_{ij} \cdot \vec{s}_j}{r_{ij}^2} w_j - w_i \psi'_i \quad (63)$$

for $i = 1, \dots, N$

where

$$\vec{r}_{ij} = (x_j - x_i)\vec{i} + (y_j - y_i)\vec{j}, \quad \vec{n}_j = y'_j\vec{i} - x'_j\vec{j} \quad \text{and} \quad \vec{s}_j = x'_j\vec{i} + y'_j\vec{j}$$

which leads to two systems of equations in matrix forms yields

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & \dots & A_{NN} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1N} \\ B_{21} & B_{22} & \dots & B_{2N} \\ \dots & \dots & \dots & \dots \\ B_{N1} & B_{N2} & \dots & B_{NN} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}$$

$$\begin{aligned}
& + \begin{pmatrix} w_1 \phi'_1 \\ w_2 \phi'_2 \\ \vdots \\ w_N \phi'_N \end{pmatrix} \quad (64) \\
\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & \dots & A_{NN} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} &= - \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1N} \\ B_{21} & B_{22} & \dots & B_{2N} \\ \dots & \dots & \dots & \dots \\ B_{N1} & B_{N2} & \dots & B_{NN} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \\
& - \begin{pmatrix} w_1 \psi'_1 \\ w_2 \psi'_2 \\ \vdots \\ w_N \psi'_N \end{pmatrix} \quad (65)
\end{aligned}$$

where

$$\begin{aligned}
A_{ij} &= \begin{cases} \frac{\bar{r}_{ij} \cdot \bar{n}_i}{r_{ij}^2} w_j = (H_{ij} y'_j - G_{ij} x'_j) w_j & j \neq i \\ -\sum_{j=1, j \neq i}^N \frac{\bar{r}_{ij} \cdot \bar{n}_i}{r_{ij}^2} w_j = -\sum_{j=1, j \neq i}^N (H_{ij} y'_j - G_{ij} x'_j) w_j & j = i \end{cases} \\
B_{ij} &= \begin{cases} \frac{\bar{r}_{ij} \cdot \bar{s}_i}{r_{ij}^2} w_j = (H_{ij} x'_j + G_{ij} y'_j) w_j & j \neq i \\ -\sum_{j=1, j \neq i}^N \frac{\bar{r}_{ij} \cdot \bar{s}_i}{r_{ij}^2} w_j = -\sum_{j=1, j \neq i}^N (H_{ij} x'_j + G_{ij} y'_j) w_j & j = i \end{cases}
\end{aligned}$$

It is advantageous to compute A_{ij} and B_{ij} for $j \neq i$ first. The coefficients A_{ii} and B_{ii} for $i = 1, \dots, N$ can then be simply obtained from

$$A_{ii} = \pi - \sum_{j=1, j \neq i}^N A_{ij}, \quad B_{ii} = - \sum_{j=1, j \neq i}^N B_{ij} \quad (66)$$

4.3 Corner Singularity for Mixed Boundary Value Problem

For simplicity, a two-corner shape as shown in Fig. 7 with Dirichlet boundary condition is used to illustrate how to use the corner point as a collocation point. Six collocation points, 1^+ , 2 , 3^+ , 3^- , 4 and 1^- , on the boundary are selected for applying integration rule. The basic idea is to derive two algebraic equations for points such as 1^+ and 1^- which are located nearby the corner point 1 first, then based on the fact that $\psi_1^- = \psi_1^+ = \psi_1$ and $\phi_1^- = \phi_1^+ = \phi_1$, these two equations can be combined into one through a limiting process, i.e. $1^+ \rightarrow 1$ and $1^- \rightarrow 1$.

Using discretized equation, Eqn. (49), for the field point 1^+ yields

$$(\psi_2 - \psi_{1^+}) \frac{\bar{r}_{1^+2} \cdot \bar{n}_2}{r_{1^+2}^2} w_2 + (\psi_3 - \psi_{1^+}) \frac{\bar{r}_{1^+3} \cdot \bar{n}_3}{r_{1^+3}^2} w_3 =$$

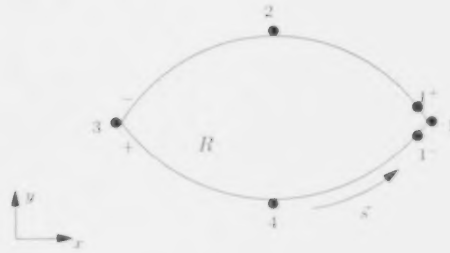


Figure 7: Two-Corner Shape

$$\begin{aligned}
 & +(\psi_3 - \psi_{1+}) \frac{\vec{r}_{1+3} \cdot \vec{n}_{3+}}{r_{1+3}^2} w_{3+} + (\psi_4 - \psi_{1+}) \frac{\vec{r}_{1+4} \cdot \vec{n}_{4+}}{r_{1+4}^2} w_{4+} \\
 & +(\psi_{1-} - \psi_{1+}) \frac{\vec{r}_{1+1-} \cdot \vec{n}_{1-}}{r_{1+1-}^2} w_{1-} \\
 = & (\phi_2 - \phi_{1+}) \frac{\vec{r}_{1+2} \cdot \vec{s}_2}{r_{1+2}^2} w_{2+} + (\phi_3 - \phi_{1+}) \frac{\vec{r}_{1+3} \cdot \vec{s}_3}{r_{1+3}^2} w_{3+} \\
 & +(\phi_3 - \phi_{1+}) \frac{\vec{r}_{1+3} \cdot \vec{s}_{3+}}{r_{1+3}^2} w_{3+} + (\phi_4 - \phi_{1+}) \frac{\vec{r}_{1+4} \cdot \vec{s}_4}{r_{1+4}^2} w_{4+} \\
 & +(\phi_{1-} - \phi_{1+}) \frac{\vec{r}_{1+1-} \cdot \vec{s}_{1-}}{r_{1+1-}^2} w_{1-} + w_{1+} \phi_{1+}' \quad (67)
 \end{aligned}$$

For the field point 1^- , a similar equation can be derived:

$$\begin{aligned}
 & (\psi_{1+} - \psi_{1-}) \frac{\vec{r}_{1-1+} \cdot \vec{n}_{1+}}{r_{1-1+}^2} w_{1+} + (\psi_2 - \psi_{1-}) \frac{\vec{r}_{1-2} \cdot \vec{n}_2}{r_{1-2}^2} w_{2-} \\
 & +(\psi_3 - \psi_{1-}) \frac{\vec{r}_{1-3} \cdot \vec{n}_{3-}}{r_{1-3}^2} w_{3-} \\
 & +(\psi_3 - \psi_{1-}) \frac{\vec{r}_{1-3} \cdot \vec{n}_{3+}}{r_{1-3}^2} w_{3+} + (\psi_4 - \psi_{1-}) \frac{\vec{r}_{1-4} \cdot \vec{n}_4}{r_{1-4}^2} w_{4-} \\
 = & (\phi_{1+} - \phi_{1-}) \frac{\vec{r}_{1-1+} \cdot \vec{s}_{1+}}{r_{1-1+}^2} w_{1+} + (\phi_2 - \phi_{1-}) \frac{\vec{r}_{1-2} \cdot \vec{s}_2}{r_{1-2}^2} w_{2-} \\
 & +(\phi_3 - \phi_{1-}) \frac{\vec{r}_{1-3} \cdot \vec{s}_{3-}}{r_{1-3}^2} w_{3-} \\
 & +(\phi_3 - \phi_{1-}) \frac{\vec{r}_{1-3} \cdot \vec{s}_{3+}}{r_{1-3}^2} w_{3+} + (\phi_4 - \phi_{1-}) \frac{\vec{r}_{1-4} \cdot \vec{s}_4}{r_{1-4}^2} w_{4-} + w_{1-} \phi_{1-}' \quad (68)
 \end{aligned}$$

Adding Eqn. (67) and Eqn. (68) together, and then taking the limit, i.e. letting

$1^+ \rightarrow 1$ and $1^- \rightarrow 1$, it follows that

$$\begin{aligned}
 & 2 \left\{ (\psi_2 - \psi_1) \frac{\vec{r}_{12} \cdot \vec{n}_2}{r_{12}^2} w_2 + (\psi_3 - \psi_1) \frac{\vec{r}_{13} \cdot (\vec{n}_3 w_{3-} + \vec{n}_3 w_{3+})}{r_{13}^2} \right. \\
 & \quad \left. + (\psi_4 - \psi_1) \frac{\vec{r}_{14} \cdot \vec{n}_4}{r_{14}^2} w_4 \right\} \\
 & + \lim_{1^+, 1^- \rightarrow 1} (\psi_{1+} - \psi_{1-}) \frac{\vec{r}_{1-1+} \cdot (\vec{n}_1 w_{1+} + \vec{n}_1 w_{1-})}{r_{1-1+}^2} \\
 = & 2 \left\{ (\phi_2 - \phi_1) \frac{\vec{r}_{12} \cdot \vec{s}_2}{r_{12}^2} w_2 + (\phi_3 - \phi_1) \frac{\vec{r}_{13} \cdot (\vec{s}_3 w_{3-} + \vec{s}_3 w_{3+})}{r_{13}^2} \right. \\
 & \quad \left. + (\phi_4 - \phi_1) \frac{\vec{r}_{14} \cdot \vec{s}_4}{r_{14}^2} w_4 \right\} \\
 & + (w_{1-} \phi_{1-}' + w_{1+} \phi_{1+}') \\
 & + \lim_{1^+, 1^- \rightarrow 1} (\phi_{1+} - \phi_{1-}) \frac{\vec{r}_{1-1+} \cdot (\vec{s}_1 w_{1+} + \vec{s}_1 w_{1-})}{r_{1-1+}^2} \quad (69)
 \end{aligned}$$



Figure 8: Corner Point

To investigate two limit terms in the above equation, we first express $(x^+ - x^-)$ and $(y^+ - y^-)$ in terms of the tangential components at t^- and t^+ .

The coordinates (x, y) of corner point t and (x^+, y^+) of the nearby point t^+ can be expressed in terms of (x^-, y^-) , the coordinates of point t^- (see Fig. 8), and tangent components of $\vec{s}_-$ and $\vec{s}_+$ as shown below,

$$\begin{aligned}
 x &= x^- + x'^- \Delta s, & x^+ &= x + x'^+ \Delta s = x^- + \Delta s(x'^- + x'^+) \\
 y &= y^- + y'^- \Delta s, & y^+ &= y + y'^+ \Delta s = y^- + \Delta s(y'^- + y'^+) \quad (70)
 \end{aligned}$$

Hence, it follows that

$$(x^+ - x^-) = \Delta s(x'^- + x'^+), \quad (y^+ - y^-) = \Delta s(y'^- + y'^+) \quad (71)$$

The following formulas are then easily derived:

$$w_{1+} \vec{n}_{1+} + w_{1-} \vec{n}_{1-} = (w_{1+} y'^+ + w_{1-} y'^-) \vec{i} - (w_{1+} x'^+ + w_{1-} x'^-) \vec{j}$$

$$\begin{aligned}
w_{1+}\vec{s}_{1+} + w_{1-}\vec{s}_{1-} &= (w_{1+}x'^{++} + w_{1-}x'^{-})\vec{i} + (w_{1+}y'^{++} + w_{1-}y'^{-})\vec{j} \\
\vec{r}_{1-1+} &= (x^+ - x^-)\vec{i} + (y^+ - y^-)\vec{j} = \Delta s[(x'^{++} + x'^{-})\vec{i} + (y'^{++} + y'^{-})\vec{j}] \\
\vec{r}_{1-1+} \cdot (w_{1+}\vec{n}_{1+} + w_{1-}\vec{n}_{1-}) &= \left\{ \Delta s[(x'^{++} + x'^{-})\vec{i} + (y'^{++} + y'^{-})\vec{j}] \right. \\
&\quad \cdot \left. \{ (w_{1+}y'^{++} + w_{1-}y'^{-})\vec{i} - (w_{1+}x'^{++} + w_{1-}x'^{-})\vec{j} \} \right\} \\
&= \Delta s(w_{1+} - w_{1-})(y'^{++}x'^{-} - x'^{++}y'^{-}) \\
&= \Delta s(w_{1+} - w_{1-}) \left\{ -(y'^{-} \cos \alpha - x'^{-} \sin \alpha)x'^{-} + (x'^{-} \cos \alpha + y'^{-} \sin \alpha)y'^{-} \right\} \\
&= \Delta s(w_{1+} - w_{1-}) \left\{ -x'^{-}y'^{-} \cos \alpha + x'^{-2} \sin \alpha + x'^{-}y'^{-} \cos \alpha + y'^{-2} \sin \alpha \right\} \\
&= \Delta s(w_{1+} - w_{1-}) \sin \alpha \\
\vec{r}_{1-1+} \cdot (w_{1+}\vec{s}_{1+} + w_{1-}\vec{s}_{1-}) &= \left\{ \Delta s[(x'^{++} + x'^{-})\vec{i} + (y'^{++} + y'^{-})\vec{j}] \right\} \\
&\quad \cdot \left\{ (w_{1+}x'^{++} + w_{1-}x'^{-})\vec{i} + (w_{1+}y'^{++} + w_{1-}y'^{-})\vec{j} \right\} \\
&= \Delta s(w_{1+} + w_{1-})(1 + x'^{++}x'^{-} + y'^{++}y'^{-}) \\
&= \Delta s w_{1+} + w_{1-} (1 - x'^{-2} \cos \alpha - x'^{-}y'^{-} \sin \alpha - y'^{-2} \cos \alpha + x'^{-}y'^{-} \sin \alpha) \\
&= \Delta s(w_{1+} + w_{1-})(1 - \cos \alpha) \\
r_{1-1+}^2 &= 2\Delta s^2(1 - \cos \alpha) \tag{72}
\end{aligned}$$

Based on the above formulas, the limits of two limit terms in Eqn. (69) are found to be:

$$\begin{aligned}
&\lim_{1+,1- \rightarrow 1} (\psi_{1+} - \psi_{1-}) \frac{\vec{r}_{1-1+} \cdot (w_{1+}\vec{n}_{1+} + w_{1-}\vec{n}_{1-})}{r_{1-1+}^2} \\
&= \lim_{1+,1- \rightarrow 1} (\psi_{1+} - \psi_{1-}) \frac{\Delta s(w_{1+} - w_{1-}) \sin \alpha}{2\Delta s^2(1 - \cos \alpha)} \\
&= \left\{ \frac{(\psi_{1+} - \psi_1)}{\Delta s} + \frac{(\psi_1 - \psi_{1-})}{\Delta s} \right\} \frac{(w_{1+} - w_{1-}) \sin \alpha}{2(1 - \cos \alpha)} \\
&= (\psi'_{1+} + \psi'_{1-}) \frac{(w_{1+} - w_{1-}) \sin \alpha}{2(1 - \cos \alpha)} \tag{73}
\end{aligned}$$

Using the relation

$$-(\psi'_{1+} + \psi'_{1-}) \frac{\sin \alpha}{1 - \cos \alpha} = (\phi'_{1+} - \phi'_{1-}) \tag{74}$$

We finally arrive

$$\lim_{1^+, 1^- \rightarrow 1} (\psi_{1^+} - \psi_{1^-}) \frac{\vec{r}_{1-1^+} \cdot (\vec{w}_{1^+} \vec{n}_{1^+} + w_{1^-} \vec{n}_{1^-})}{r_{1-1^+}^2} = -(\phi'_{1^+} - \phi'_{1^-}) \frac{w_{1^+} - w_{1^-}}{2} \quad (75)$$

We now turn to the other limit term:

$$\begin{aligned} & \lim_{1^+, 1^- \rightarrow 1} (\phi_{1^+} - \phi_{1^-}) \frac{\vec{r}_{1-1^+} \cdot (w_{1^+} \vec{s}_{1^+} + w_{1^-} \vec{s}_{1^-})}{r_{1-1^+}^2} w_1 \\ &= \lim_{1^+, 1^- \rightarrow 1} (\phi_{1^+} - \phi_{1^-}) \frac{\Delta s (w_{1^-} + w_{1^+}) (1 - \cos \alpha)}{2 \Delta s^2 (1 - \cos \alpha)} \\ &= \left\{ \frac{(\phi_{1^+} - \phi_{1^-})}{\Delta s} + \frac{\phi_{1^-} - \phi_{1^+}}{\Delta s} \right\} \frac{(w_{1^-} + w_{1^+})}{2} \\ &= -(\phi'_{1^+} - \phi'_{1^-}) \frac{(w_{1^-} - w_{1^+})}{2} \quad (76) \end{aligned}$$

Substituting Eqn. (76) back to Eqn. (69) gives the combined equation for the corner point 1:

$$\begin{aligned} & (\psi_2 - \psi_1) \frac{\vec{r}_{12} \cdot \vec{n}_2}{r_{12}^2} w_2 + (\psi_3 - \psi_1) \frac{\vec{r}_{13} \cdot (w_{3^+} \vec{n}_{3^+} + w_{3^-} \vec{n}_{3^-})}{r_{13}^2} \\ & + (\psi_4 - \psi_1) \frac{\vec{r}_{14} \cdot \vec{n}_4}{r_{14}^2} w_4 \\ &= (\phi_2 - \phi_1) \frac{\vec{r}_{12} \cdot \vec{s}_2}{r_{12}^2} w_2 + (\phi_3 - \phi_1) \frac{\vec{r}_{13} \cdot (w_{3^+} \vec{s}_{3^+} + w_{3^-} \vec{s}_{3^-})}{r_{13}^2} \\ & + (\phi_4 - \phi_1) \frac{\vec{r}_{14} \cdot \vec{s}_4}{r_{14}^2} w_4 + (\phi'_{1^+} + \phi'_{1^-}) \frac{(w_{1^-} + w_{1^+})}{2} \\ & + (\phi'_{1^+} - \phi'_{1^-}) \frac{(w_{1^-} - w_{1^+})}{2} \\ &= (\phi_2 - \phi_1) \frac{\vec{r}_{12} \cdot \vec{s}_2}{r_{12}^2} w_2 + (\phi_3 - \phi_1) \frac{\vec{r}_{13} \cdot (w_{3^+} \vec{s}_{3^+} + w_{3^-} \vec{s}_{3^-})}{r_{13}^2} \\ & + (\phi_4 - \phi_1) \frac{\vec{r}_{14} \cdot \vec{s}_4}{r_{14}^2} w_4 + w_{1^+} \phi'_{1^+} + w_{1^-} \phi'_{1^-} \quad (77) \end{aligned}$$

Based on the same idea to derive equations for field points 2, 3 and 4, the following system is obtained to solve for $\psi_j, j = 1, \dots, 4$:

$$\begin{pmatrix} -\sum_{j=1, j \neq 1}^4 A_{1,j} & \frac{\phi_{12} \cdot \vec{s}_2}{r_{12}^2} w_2 & \frac{\phi_{13} \cdot (w_{3^+} \vec{s}_{3^+} + w_{3^-} \vec{s}_{3^-})}{r_{13}^2} & \frac{\phi_{14} \cdot \vec{s}_4}{r_{14}^2} w_4 \\ \frac{\phi_{21} \cdot (w_{1^+} \vec{s}_{1^+} + w_{1^-} \vec{s}_{1^-})}{r_{21}^2} & -\sum_{j=1, j \neq 2}^4 A_{2,j} & \frac{\phi_{23} \cdot (w_{3^+} \vec{s}_{3^+} + w_{3^-} \vec{s}_{3^-})}{r_{23}^2} & \frac{\phi_{24} \cdot \vec{s}_4}{r_{24}^2} w_4 \\ \frac{\phi_{31} \cdot (w_{1^+} \vec{s}_{1^+} + w_{1^-} \vec{s}_{1^-})}{r_{31}^2} & \frac{\phi_{32} \cdot \vec{s}_2}{r_{32}^2} w_2 & -\sum_{j=1, j \neq 3}^4 A_{3,j} & \frac{\phi_{34} \cdot \vec{s}_4}{r_{34}^2} w_4 \\ \frac{\phi_{41} \cdot (w_{1^+} \vec{s}_{1^+} + w_{1^-} \vec{s}_{1^-})}{r_{41}^2} & \frac{\phi_{42} \cdot \vec{s}_2}{r_{42}^2} w_2 & \frac{\phi_{43} \cdot (w_{3^+} \vec{s}_{3^+} + w_{3^-} \vec{s}_{3^-})}{r_{43}^2} & -\sum_{j=1, j \neq 4}^4 A_{4,j} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} -\sum_{j=1, j \neq 1}^4 B_{1,j} & \frac{\phi_{12} \phi_2 w_2}{\tau_{12}^2} & \frac{\phi_{13} (w_2 + \phi_{21} + w_2 - \phi_{2-})}{\tau_{13}^2} & \frac{\phi_{14} \phi_4 w_4}{\tau_{14}^2} \\ \frac{\phi_{21} (w_1 + \phi_{1+} + w_1 - \phi_{1-})}{\tau_{21}^2} & -\sum_{j=1, j \neq 2}^4 B_{2,j} & \frac{\phi_{23} (w_2 + \phi_{2+} + w_2 - \phi_{2-})}{\tau_{23}^2} & \frac{\phi_{24} \phi_4 w_4}{\tau_{24}^2} \\ \frac{\phi_{31} (w_1 + \phi_{1+} + w_1 - \phi_{1-})}{\tau_{31}^2} & \frac{\phi_{32} \phi_2 w_2}{\tau_{32}^2} & -\sum_{j=1, j \neq 3}^4 B_{3,j} & \frac{\phi_{34} \phi_4 w_4}{\tau_{34}^2} \\ \frac{\phi_{41} (w_1 + \phi_{1+} + w_1 - \phi_{1-})}{\tau_{41}^2} & \frac{\phi_{42} \phi_2 w_2}{\tau_{42}^2} & \frac{\phi_{43} (w_2 + \phi_{2+} + w_2 - \phi_{2-})}{\tau_{43}^2} & -\sum_{j=1, j \neq 4}^4 B_{4,j} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \\
&+ \begin{pmatrix} w_1 \phi'_{1+} + w_1 - \phi'_{1-} \\ w_2 \phi'_2 \\ w_3 \phi'_{3+} + w_3 - \phi'_{3-} \\ w_4 \phi'_4 \end{pmatrix} \quad (78)
\end{aligned}$$

Two sets of equations can be written as follows:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} + \begin{pmatrix} w_1 \phi'_{1+} + w_1 - \phi'_{1-} \\ w_2 \phi'_2 \\ w_3 \phi'_{3+} + w_3 - \phi'_{3-} \\ w_4 \phi'_4 \end{pmatrix} \quad (79)$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = - \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} - \begin{pmatrix} w_1 \psi'_{1+} + w_1 - \psi'_{1-} \\ w_2 \psi'_2 \\ w_3 \psi'_{3+} + w_3 - \psi'_{3-} \\ w_4 \psi'_4 \end{pmatrix} \quad (80)$$

If we consider a mixed boundary condition, ψ'_{3+} , ψ'_4 , ψ'_{1-} can be directly calculated.

The eight unknowns in the above sets of equations are then $\psi_2, \phi_4, \phi'_1, \phi'_3, \phi'_4, \psi'_1, \psi'_2, \psi'_3$. To solve unknowns, we can use Eqns. 2, and 8 to solve for ψ_2 and ϕ_4 first, then based on Eqns. 1, 3, 4, 5 and 6, 7 to calculate $\phi'_1, \phi'_3, \phi'_4, \psi'_1, \psi'_2$ and ψ'_3 .

5 Euler-Lagrangian Method

Euler-Lagrangian method is a useful tool to investigate the motion of an unsteady velocity field. In an Eulerian space an observer watches the world evolving around him at a fixed geographical point. In a Lagrangian space an observer watches the world evolving around him as he travels with a fluid particle. In this research, those two spaces are used at same time to explore the motion of fluid particles along the boundary. The boundary integral equation is solved in Eulerian space first and the new boundary is then determined by tracing the Lagrangian particles in Lagrangian space.

In Lagrangian space, the kinematic condition on the free surface is described as:

$$\frac{Dz}{Dt} = u + iv \quad (81)$$

and dynamic free-surface condition is written in the form of:

$$\frac{D\phi}{Dt} = \frac{1}{2}(u^2 + v^2) - g\eta \quad (82)$$

Based on desingularized Cauchy's formula, the procedure of Euler-Lagrangian method is summarized as follows:

1. Solving the stream function ψ on the free surface with a given boundary condition, say $\phi = 0$ on the free surface at $t = 0$.
2. Computing velocity components u and v on the free surface.
3. Based on Eqn. (81), tracing the fluid particles to find the new free surface for next time step.
4. Based on Eqn. (82), updating the velocity potential for next time step.
5. Repeat the process for next time step.

For the comparison purpose, in our numerical computation, three approaches to carry the integration in time, namely, Euler, second order predictor-corrector and Runge-Kutta methods are used. The numerical results of transient waves in a container from these three approaches are illustrated in Section 7.

6 Formulation of B-V Problem for $\partial\phi/\partial t$ and $\partial\psi/\partial t$

To calculate the pressure on the body surface, the Bernoulli's equation shown in Eqn. (83) is used.

$$\frac{p}{\rho} = -\frac{\partial\phi}{\partial t} - \frac{1}{2}(u^2 + v^2) - gy \quad (83)$$

The velocity of fluid particles u and v can be solved from boundary integral equation. A novel numerical method to solve $\frac{\partial\phi}{\partial t}$ on the body surface is developed in this section.

Since $f(z) = \phi + i\psi$ is an analytical function in the domain of interest at any moment, the function $\frac{\partial f(z)}{\partial t}$ is also analytical anywhere. Thus, the Cauchy's formula can be applied to solve $\frac{\partial\phi}{\partial t}$ and $\frac{\partial\psi}{\partial t}$ on boundary.

Assuming the velocity of the body is $-V(t)\vec{j}$, the body surface condition is written as

$$\frac{\partial\phi}{\partial n} = \frac{\partial\psi}{\partial s} = -V(t)\vec{j} \cdot (s_y\vec{i} - s_x\vec{j}) = V(t)\frac{\partial x}{\partial s} \quad (84)$$

Accordingly,

$$\psi|_{s_B} = V(t)x \quad \frac{\partial\psi}{\partial t} = \frac{dV}{dt}x \quad (85)$$

On the free surface, we have

$$\frac{\partial\phi}{\partial t} = -\frac{1}{2}(u^2 + v^2) - gy \quad (86)$$

Therefore, the boundary value problem is described as

1. On the body surface, $\partial\psi/\partial t$ is given.
2. On the free surface, $\partial\phi/\partial t$ is prescribed.
3. Cauchy's formula can be applied to solve this mixed boundary value problem.

$$\pi i f(z) = - \int_{-\infty}^{\infty} \frac{f(z_0)}{z_0 - z} \frac{dz_0}{ds} \quad (87)$$

where

$$f(z) = \frac{\partial \phi}{\partial t} + i \frac{\partial \psi}{\partial t}$$

Based on this formulation, the numerical computation of $\frac{\partial \phi}{\partial t}$ is verified in section 7 for linear container problem. The numerical results agree with analytical solution.

7 Examples

7.1 Linear Container Problem



Figure 9: Container

Linear container problem is a simple and good example to verify our numerical scheme for solving time-domain nonlinear wave problem. Due to the geometry of the container, it can be also used to validate the numerical treatment of mixed boundary problem and corner singularity developed in Section 4. For each time step, the new grid points on the free surface for solving boundary value problem are found from NURBS curve fitting.

The initial wave profile is given by

$$y(x, t = 0) = a \cos\left(\frac{2\pi x}{b}\right) \quad (88)$$

In the numerical computation, 20 Gaussian points are allocated on each boundary. On the free surface, Labatto quadrature which includes the endpoints of the interval $[-1, 1]$ shown below is used:

$$\int_{-1}^1 f(x) dx = w_1 f(-1) + w_n f(1) + \sum_{i=2}^{n-1} w_i f(x_i) \quad (89)$$

where the free abscissas x_i for $i = 2, 3, \dots, n-1$ are the roots of the Polynomial $P'_{n-1}(x)$. Notice that $P(x)$ is a Legendre polynomial.

Therefore, the intersection points, $(0, 0)$ and $(b, 0)$, between free-surface and body surface can be used as collocation points. The numerical treatment of these two

points is similar to that used on water entry problem. In order to avoid the corner singularities at point $(0, -d)$ and point (b, d) , Legendre Gaussian quadrature is applied to the other three boundaries. This is due to the fact that these two corner points can not affect the free surface flow. Thus, 80 collocation points are used in the computation. It is worthwhile to mention that at the intersection points $(0, 0)$ and $(b, 0)$ both free-surface and body surface conditions are satisfied exactly in the numerical computation. Fig. 10 shows the grid distribution on the boundary of the container.

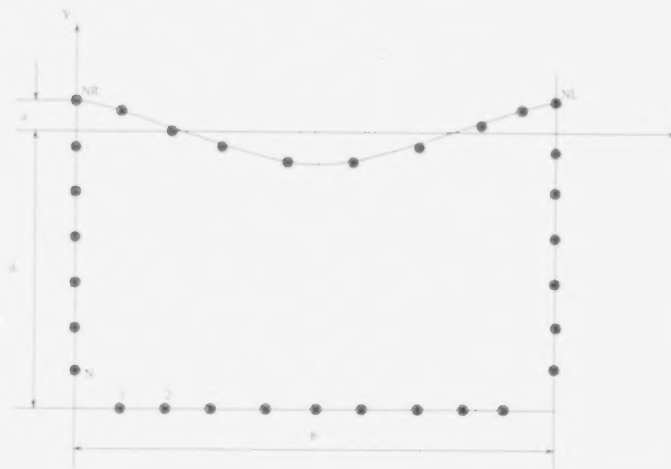


Figure 10: Nodal Points on the Boundary

7.1.1 Governing Equation and Boundary/Initial Conditions

1. Governing Equation

$$\nabla^2 \phi = 0 \quad (90)$$

2. Boundary Conditions

$$\frac{\partial \phi}{\partial y}(x, -d, t) = 0, \quad \frac{\partial \phi}{\partial x}(0, y, t) = 0, \quad \frac{\partial \phi}{\partial x}(b, y, t) = 0 \quad (91)$$

3. Initial Condition

$$\phi(x, y, 0) = 0, \quad \frac{\partial \phi}{\partial t}(x, 0, 0) = -g\eta(x, 0), \quad \eta(x, 0) = a \cos \frac{2\pi x}{b} \quad (92)$$

7.1.2 Numerical Solution

In the numerical computation, three different time integrations, namely, Euler, second order predictor-corrector and Runge-Kutta methods are used for the comparison purpose. They are described as follows:

1. Euler Method

- (a) Calculate $\eta(x, t + \Delta t)$ from

$$\eta(x, t + \Delta t) = \eta(x, t) + \frac{\partial \phi}{\partial y}(x, 0, t) \Delta t = \eta(x, t) - \frac{\partial \psi}{\partial x}(x, 0, t) \Delta t \quad (93)$$

Since

$$\frac{\partial \eta}{\partial t}(x, t) = \frac{\partial \phi}{\partial y}(x, 0, t) = -\frac{\partial \psi}{\partial x}(x, 0, t) \quad (94)$$

- (b) Find $\phi(x, 0, t + \Delta t)$

$$\phi(x, 0, t + \Delta t) = \phi(x, 0, t) - g\eta(x, t) \Delta t \quad (95)$$

This is due to the fact that

$$\frac{\partial \phi}{\partial t}(x, 0, t) = -g\eta(x, t) \quad (96)$$

- (c) Solve the boundary value problem to find $\partial \phi / \partial y(x, 0, t + \Delta t)$ or $\psi(x, 0, t + \Delta t)$
 (d) Use curve fitting to find $\partial \psi / \partial x(x, 0, t + \Delta t)$
 (e) Repeat the process for $t = 2\Delta t, 3\Delta t, \dots, N\Delta t$

The numerical results of the free surface elevation at the center of the container is shown in Fig. 11 for $\Delta t = 0.0005$.

2. Second Order Predictor-Corrector Method

- (a) Predictor

- i. Calculate $\eta^{(p)}(x, t + \Delta t)$ from

$$\begin{aligned} \eta^{(p)}(x, t + \Delta t) &= \eta(x, t) + \frac{\partial \phi}{\partial y}(x, 0, t) \Delta t \\ &= \eta(x, t) - \frac{\partial \psi}{\partial x}(x, 0, t) \Delta t \end{aligned} \quad (97)$$

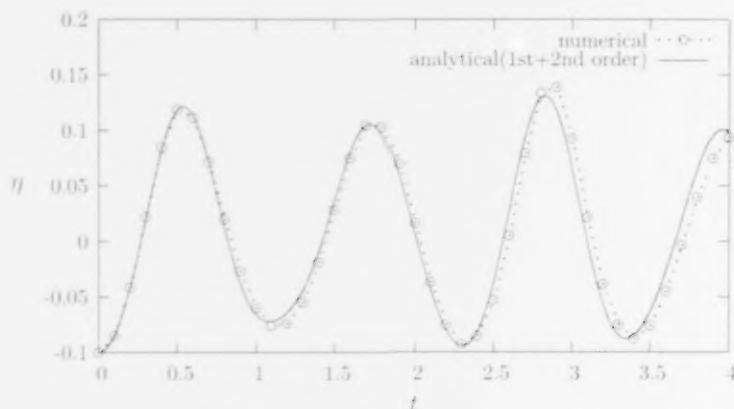


Figure 11: Time history of elevation at the center point of a container: $u/D = 0.1$, Euler method, $\Delta t = 0.0005s$

ii. Find $\phi^{(p)}(x, 0, t + \Delta t)$

$$\phi^{(p)}(x, 0, t + \Delta t) = \phi(x, 0, t) - g\eta(x, t)\Delta t \quad (98)$$

iii. Solve the boundary value problem to find $\partial\phi/\partial y^{(p)}(x, 0, t + \Delta t)$ or $\psi^{(p)}(x, 0, t + \Delta t)$

iv. Use curve fitting to find $\partial\psi/\partial x^{(p)}(x, 0, t + \Delta t)$

(b) Corrector

i. Calculate $\eta(x, t + \Delta t)$ from

$$\begin{aligned} \eta(x, t + \Delta t) &= \eta(x, t) \\ &\quad + \frac{1}{2} \left\{ \frac{\partial\phi}{\partial y}(x, 0, t) + \frac{\partial\phi^{(p)}}{\partial y}(x, 0, t + \Delta t) \right\} \Delta t \\ &= \eta(x, t) \\ &\quad - \frac{1}{2} \left\{ \frac{\partial\psi}{\partial x}(x, 0, t) + \frac{\partial\psi^{(p)}}{\partial x}(x, 0, t + \Delta t) \right\} \Delta t \quad (99) \end{aligned}$$

ii. Find $\phi(x, 0, t + \Delta t)$

$$\phi(x, 0, t + \Delta t) = \phi(x, 0, t) - \frac{1}{2}g \left\{ \eta(x, t) + \eta^{(p)}(x, t + \Delta t) \right\} \Delta t \quad (100)$$

- iii. Solve the boundary value problem to find $\partial\phi/\partial y(x, 0, t+\Delta t)$ or $\psi(x, 0, t+\Delta t)$
- iv. Use curve fitting to find $\partial\psi/\partial x(x, 0, t+\Delta t)$
- (c) Repeat the process for $t = 2\Delta t, 3\Delta t, \dots, N\Delta t$

Fig. 12 shows results for the free surface elevation at subsequent times at the center of the container based on $\Delta t = 0.001$ and second order method.

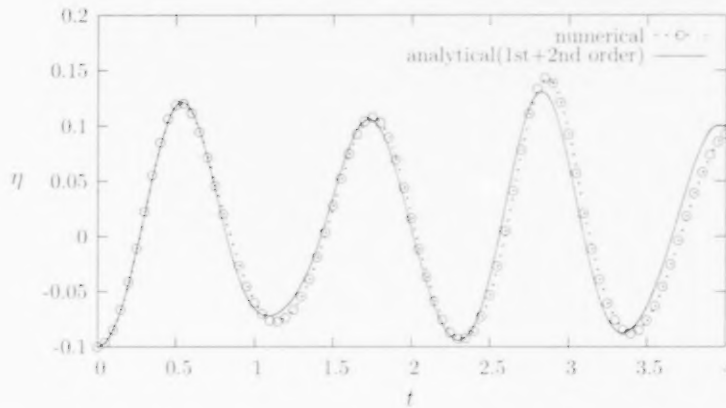


Figure 12: Time history of elevation at the center point of a container; $a/D = 0.1$, Second-order Predictor-Corrector method, $\Delta t = 0.001$ s

3. Runge-Kutta Method

- (a) find k_1 and q_1

$$k_1 = -g\eta(x, t)\Delta t, \quad q_1 = -\frac{\partial\psi}{\partial x}(x, 0, t)\Delta t \quad (101)$$

- (b) find k_2

$$k_2 = -g\left\{\eta(x, t) + \frac{1}{2}q_1\right\}\Delta t \quad (102)$$

- (c) solve boundary value problem to find $\partial\psi/\partial x^{(2)}(x, 0, t+0.5\Delta t)$. The velocity potential on the free surface is given by

$$\phi(x, 0, t+0.5\Delta t) = \phi(x, 0, t) + \frac{k_1}{2} = \phi(x, 0, t) - g\eta(x, t) \times 0.5\Delta t \quad (103)$$

(d) find q_2 ,

$$q_2 = -\frac{\partial \psi^{(2)}}{\partial x}(x, 0, t + 0.5\Delta t)\Delta t \quad (104)$$

(e) find k_3

$$k_3 = -g \left\{ \eta(x, t) + \frac{1}{2}q_2 \right\} \Delta t \quad (105)$$

(f) solve boundary value problem to find $\partial \psi / \partial x^{(3)}(x, 0, t + 0.5\Delta t)$. The velocity potential on free surface is given by

$$\phi(x, 0, t + 0.5\Delta t) = \phi(x, 0, t) + \frac{k_2}{2} \quad (106)$$

(g) find q_3 ,

$$q_3 = -\frac{\partial \psi^{(3)}}{\partial x}(x, 0, t + 0.5\Delta t)\Delta t \quad (107)$$

(h) find k_4

$$k_4 = -g \left\{ \eta(x, t) + q_3 \right\} \Delta t \quad (108)$$

(i) solve boundary value problem to find $\partial \psi / \partial x^{(4)}(x, 0, t + \Delta t)$. The velocity potential on free surface is given by

$$\phi(x, 0, t + \Delta t) = \phi(x, 0, t) + k_3 \quad (109)$$

(j) find q_4 ,

$$q_4 = -\frac{\partial \psi^{(4)}}{\partial x}(x, 0, t + \Delta t)\Delta t \quad (110)$$

(k) Final equation

$$\begin{aligned} \phi(x, 0, t + \Delta t) &= \phi(x, 0, t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ \eta(x, 0, t + \Delta t) &= \eta(x, 0, t) + \frac{1}{6}(q_1 + 2q_2 + 2q_3 + q_4) \end{aligned} \quad (111)$$

(l) solve boundary value problem to find $\partial \psi / \partial x(x, 0, t + \Delta t)$ for the next time step

Using $\Delta t = 0.005$ the free surface elevation at the center of the container is depicted in Fig. 13.

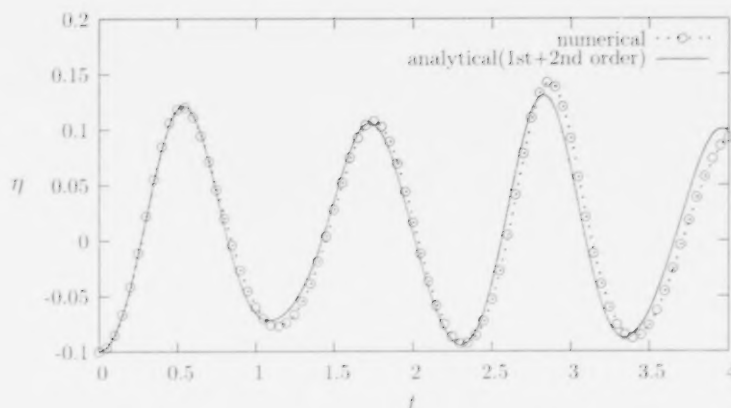


Figure 13: Time history of elevation at the center point of a container: $a/D = 0.1$, Runge-Kutta method, $\Delta t = 0.005s$

From Fig. 11 to Fig. 13, it can be seen that high order Runge-Kutta method gives fairly good accuracy with large time step. Furthermore, it also shows that in the beginning the agreement is excellent. Numerical solution departs from analytical one when time increases, which might be due to the neglect of higher order terms in t in the analytical solution.

7.1.3 Distribution of $\partial\phi/\partial t$ on container wall

To verify the numerical scheme developed in Section 6 for solving $\partial\phi/\partial t$ on body surface, the comparison with the linear analytical solution (in Appendix B) are made. Fig. 14 shows the comparison of the numerical results and the linear analytical solution at the center bottom of container. They are in good agreement.

7.2 Wedge Entry Problem

Fig. 15 shows the collocation points used in the computation for the wedge entry problem.

The potentials, $\phi_1 \cdots \phi_{NL}$, $\phi_{NR} \cdots \phi_N$, are given on the free surface S_F . Two points, NL and NR , are two intersection points between body surface and free surface. From the given ϕ 's, ϕ' can be easily found by curve fitting on S_F . The

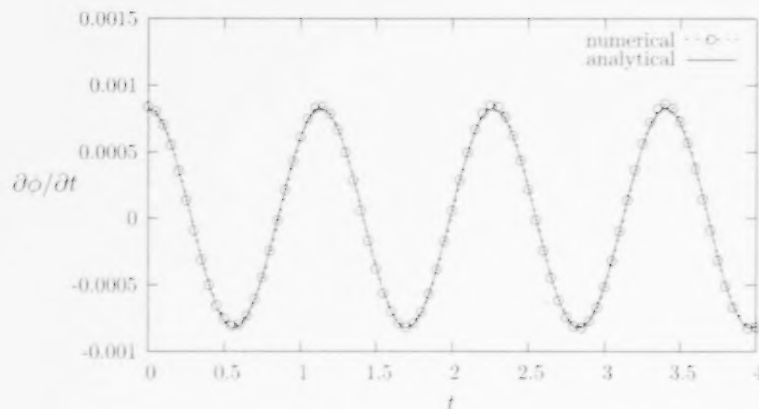


Figure 14: Time history of $\partial\phi/\partial t$ on container wall: $a/D = 0.001$



Figure 15: Grid Points of Wedge Entry Problem

stream function, $\psi_{NL} \cdots \psi_{NR}$, and the corresponding ψ' 's are given on body surface S_B . Based on the numerical scheme developed in Section 4.3, the system of algebraic equations to solve for this mixed boundary value problem is written as

$$\begin{pmatrix}
 A_{11} & A_{1NL} & -B_{1NL+1} & -B_{1NR-1} & A_{1NR} & A_{1N} \\
 A_{NL1} & A_{NLNL} & -B_{NLNL+1} & -B_{NLNR-1} & A_{NLNR} & A_{NLN} \\
 B_{NL+11} & B_{NL+1NL} & A_{NL+1NL+1} & A_{NL+1NR-1} & B_{NL+1NR} & B_{NL+1N} \\
 B_{NR-11} & B_{NR-1NL} & A_{NR-1NL+1} & A_{NR-1NR-1} & B_{NR-1NR} & B_{NR-1N} \\
 A_{NR1} & A_{NRNL} & -B_{NRNL+1} & -B_{NRNR-1} & A_{NRNR} & A_{NRN} \\
 A_{N1} & A_{NNL} & -B_{NNL+1} & -B_{NNR-1} & A_{NNNR} & A_{NN}
 \end{pmatrix}
 \times
 \begin{pmatrix}
 \psi_1 \\
 \psi_{NL} \\
 \psi_{NL+1} \\
 \psi_{NR-1} \\
 \psi_{NR} \\
 \psi_N
 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} B_{11} & \cdots & B_{1NL} & A_{1NL+1} & -A_{1NR-1} & B_{1NR} & \cdots & B_{1N} \\ B_{NL1} & \cdots & B_{NLSL} & A_{NLSL+1} & -A_{NLNR-1} & B_{NLNR} & \cdots & B_{NLSN} \\ -A_{NL+11} & \cdots & -A_{NL+1NL} & -B_{NL+1NL+1} & -B_{NL+1NR-1} & -A_{NL+1NR} & \cdots & -A_{NL+1N} \\ -A_{NR-11} & \cdots & -A_{NR-1NL} & -B_{NR-1NL+1} & -B_{NR-1NR-1} & -A_{NR-1NR} & \cdots & -A_{NR-1N} \\ B_{NR1} & \cdots & B_{NRSL} & A_{NRSL+1} & -A_{NRNR-1} & B_{NRNR} & \cdots & B_{NRN} \\ B_{N1} & \cdots & B_{NLSL} & A_{NLSL+1} & -A_{NNR-1} & B_{NNR} & \cdots & B_{NN} \end{pmatrix} \\
&\times \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{NL-1} \\ \phi_{NL+1} \\ \vdots \\ \phi_{NR-1} \\ \phi_{NR} \\ \vdots \\ \phi_N \end{pmatrix} + \begin{pmatrix} w_1 \phi_1^+ \\ w_{NL}(\phi_{NL}^{t-} + \phi_{NL}^{t+}) \\ w_{NL+1} \phi_{NL+1}^+ \\ \vdots \\ w_{NR-1} \phi_{NR-1}^+ \\ w_{NR}(\phi_{NR}^{t-} + \phi_{NR}^{t+}) \\ \vdots \\ w_N \phi_N^+ \end{pmatrix} \quad (112)
\end{aligned}$$

where $\phi_{NL}^{t+}, \phi_{NL}^{t-}, \phi_{NR}^{t+}$ and ϕ_{NR}^{t-} are due to the discontinuities of ϕ^t across the corner point.

It should be noted that the intersection points, NL and NR , are common to both the wetted body surface and the free-surface, and the stream functions at these two points ψ_{NL}, ψ_{NR} are regarded as known in terms of their boundary conditions. Therefore, the terms associated with ψ_{NL}, ψ_{NR} can be moved to the right-hand side of the equations and the NL and the NR rows can be eliminated from the system. Moreover, the unknown $\phi_{NL}^{t+}, \phi_{NL}^{t-}, \phi_{NR}^{t+}$ and ϕ_{NR}^{t-} can also be discarded from the equations. Therefore, the system of equations can be written as follows:

$$\begin{aligned}
&\begin{pmatrix} A_{11} & \cdots & A_{1NL-1} & -B_{1NL+1} & -B_{1NR-1} & A_{1NR+1} & \cdots & A_{1N} \\ A_{NL-11} & \cdots & A_{NL-1NL-1} & -B_{NL-1NL+1} & -B_{NL-1NR-1} & A_{NL-1NR+1} & \cdots & A_{NL-1N} \\ B_{NL+11} & \cdots & B_{NL+1NL-1} & A_{NL+1NL+1} & -A_{NL+1NR-1} & B_{NL+1NR+1} & \cdots & B_{NL+1N} \\ B_{NR-11} & \cdots & B_{NR-1NL-1} & A_{NR-1NL+1} & -A_{NR-1NR-1} & B_{NR-1NR+1} & \cdots & B_{NR-1N} \\ A_{NR+11} & \cdots & A_{NR+1NL-1} & -B_{NR+1NL+1} & -B_{NR+1NR-1} & A_{NR+1NR+1} & \cdots & A_{NR+1N} \\ A_{N1} & \cdots & A_{NLSL-1} & -B_{NLSL+1} & -B_{NNR-1} & A_{NNR+1} & \cdots & A_{NN} \end{pmatrix} \\
&\times \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{NL-1} \\ \phi_{NL+1} \\ \vdots \\ \phi_{NR-1} \\ \phi_{NR+1} \\ \vdots \\ \phi_N \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} B_{11} & B_{1NL-1} & A_{1NL+1} & -A_{1NR-1} & B_{1NR+1} & B_{1N} \\ B_{NL-11} & B_{NL-1NL-1} & A_{NL-1NL+1} & A_{NL-1NR-1} & B_{NL-1NR+1} & B_{NL-1N} \\ -A_{NL+11} & -A_{NL+1NL-1} & -B_{NL+1NL+1} & -B_{NL+1NR-1} & -A_{NL+1NR+1} & -A_{NL+1N} \\ -A_{NR-11} & -A_{NR-1NL-1} & -B_{NR-1NL+1} & -B_{NR-1NR-1} & -A_{NR-1NR+1} & -A_{NR-1N} \\ B_{NR+11} & B_{NR+1NL-1} & A_{NR+1NL+1} & A_{NR+1NR-1} & B_{NR+1NR+1} & B_{NR+1N} \end{pmatrix} \\
&\times \begin{pmatrix} \phi_1 \\ \phi_{NL-1} \\ \phi_{NL+1} \\ \phi_{NR-1} \\ \phi_{NR+1} \\ \phi_N \end{pmatrix} + \begin{pmatrix} -\phi_1 \phi_1^* \\ \phi_{NL-1} \phi_{NL-1}^* \\ \phi_{NL+1} \phi_{NL+1}^* \\ \phi_{NR-1} \phi_{NR-1}^* \\ \phi_{NR+1} \phi_{NR+1}^* \\ \phi_N \phi_N^* \end{pmatrix} \\
&+ \begin{pmatrix} B_{1NL} \phi_{NL} + B_{1NR} \phi_{NR} - A_{1NL} \phi_{NL} - A_{1NR} \phi_{NR} \\ B_{NL-1NL} \phi_{NL} + B_{NL-1NR} \phi_{NR} - A_{NL-1NL} \phi_{NL} - A_{NL-1NR} \phi_{NR} \\ -A_{NL+1NL} \phi_{NL} - A_{NL+1NR} \phi_{NR} - B_{NL+1NL} \phi_{NL} - B_{NL+1NR} \phi_{NR} \\ -A_{NR-1NL} \phi_{NL} - A_{NR-1NR} \phi_{NR} - B_{NR-1NL} \phi_{NL} - B_{NR-1NR} \phi_{NR} \\ B_{NR+1NL} \phi_{NL} + B_{NR+1NR} \phi_{NR} - A_{NR+1NL} \phi_{NL} - A_{NR+1NR} \phi_{NR} \\ B_{NN} \phi_{NL} + B_{NN} \phi_{NR} - A_{NN} \phi_{NL} - A_{NN} \phi_{NR} \end{pmatrix} \quad (113)
\end{aligned}$$

where

$$\begin{aligned}
A_{ij} &= \begin{cases} \frac{\vec{r}_{ij} \cdot \vec{n}_i}{r_{ij}^2} w_j, & j \neq i, j \neq NL \text{ or } NR \\ \frac{\vec{r}_{ij} \cdot (\vec{n}_i^+ + \vec{n}_i^-)}{r_{ij}^2} w_j, & j = NL, NR \\ \pi - \sum_{j=1, j \neq i}^N A_{ij} & j = i \end{cases} \\
B_{ij} &= \begin{cases} \frac{\vec{r}_{ij} \cdot \vec{s}_i}{r_{ij}^2} w_j, & j \neq i, j \neq NL \text{ or } NR \\ \frac{\vec{r}_{ij} \cdot (\vec{s}_i^+ + \vec{s}_i^-)}{r_{ij}^2} w_j, & j = NL, NR \\ -\sum_{j=1, j \neq i}^N B_{ij} & j = i \end{cases}
\end{aligned}$$

$$\vec{r}_{ij} = (x_j - x_i)\vec{i} + (y_j - y_i)\vec{j}, \quad \vec{n}_i = y_i'\vec{i} - x_i'\vec{j}, \quad \vec{s}_i = x_i'\vec{i} + y_i'\vec{j}$$

At point NL and point NR , it follows that

$$\vec{n}_j = y_j'\vec{i} - x_j'\vec{j}, \quad \vec{n}_j^+ = y_j^{'+}\vec{i} - x_j^{'+}\vec{j}, \quad \vec{s}_j = x_j'\vec{i} + y_j'\vec{j}, \quad \vec{s}_j^+ = x_j^{'+}\vec{i} + y_j^{'+}\vec{j}$$

After solving for the potential ϕ and stream function ψ , the unknown ϕ' and ψ' along the entire boundary can be calculated. Actually ϕ' is the tangential velocity of fluid particles and ψ' is the normal velocity of fluid particles, thus u, v of particles can be found from

$$u = \phi' \cos \theta + \psi' \sin \theta \quad (114)$$

$$v = \phi' \sin \theta - \psi' \cos \theta \quad (115)$$

where $\theta = \cos^{-1} x'$

For the velocity at intersection points between free-surface and body surface, the ϕ' , tangential velocity along boundary surface, is found from the distribution of ϕ on body surface because we assume the particles at the intersection points do not leave the body surface.

It has been shown that Runge-Kutta method can give good accuracy from numerical experiment of container problem so that this method is applied to march forward in time for this wedge entry problem. After new positions of the particles are determined from Euler-Lagrangian method, the curve fitting is applied to find the new surface profile from which x' , y' and new Gaussian points along entire surface boundary are calculated for next time step. The whole process is repeated until the desired time is reached.

Fig. 16. shows the numerical results of the jet flow for a wedge of deadrise angle $\alpha = 60^\circ$ entering the water. Starting from the initial condition with just the wedge vertex in calm water, the free surface quickly becomes self-similar and agrees very well with the theoretical result from Dobrovol'skaya [7].

Fig. 17, Fig. 18, Fig. 19 and Fig. 20 shows the comparison of self-similar flow and pressure distribution on wetted surface of wedge with the results of Dobrovol'skaya[7] for wedge of deadrise angles 60° and 45° . It shows the agreement. Fig. 21 shows a comparison of slamming coefficients predicted by SHIPMO [18] for wedge sections based on Stavovy and Chuang's method and our numerical results.

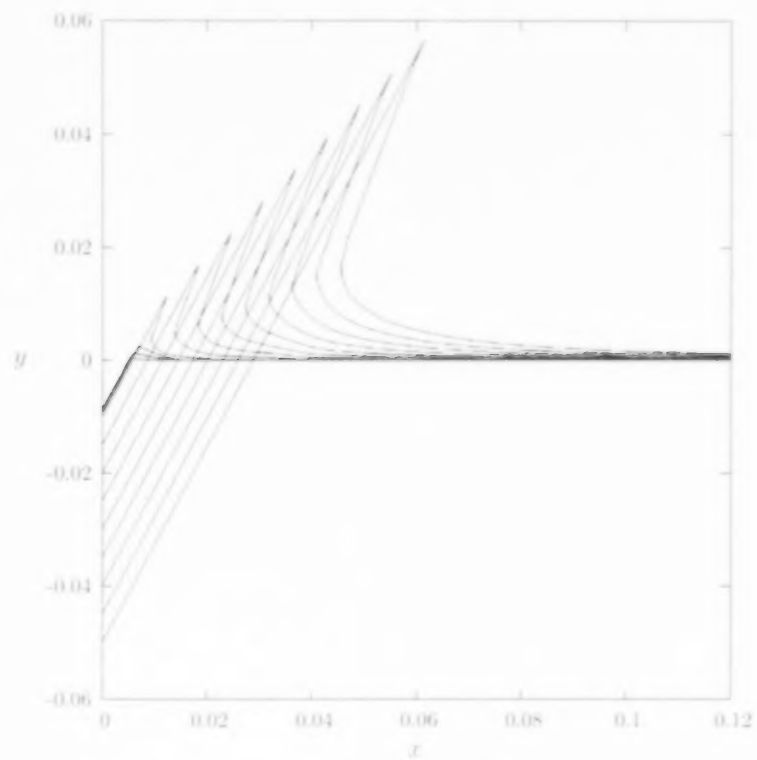


Figure 16: Development of jet flow

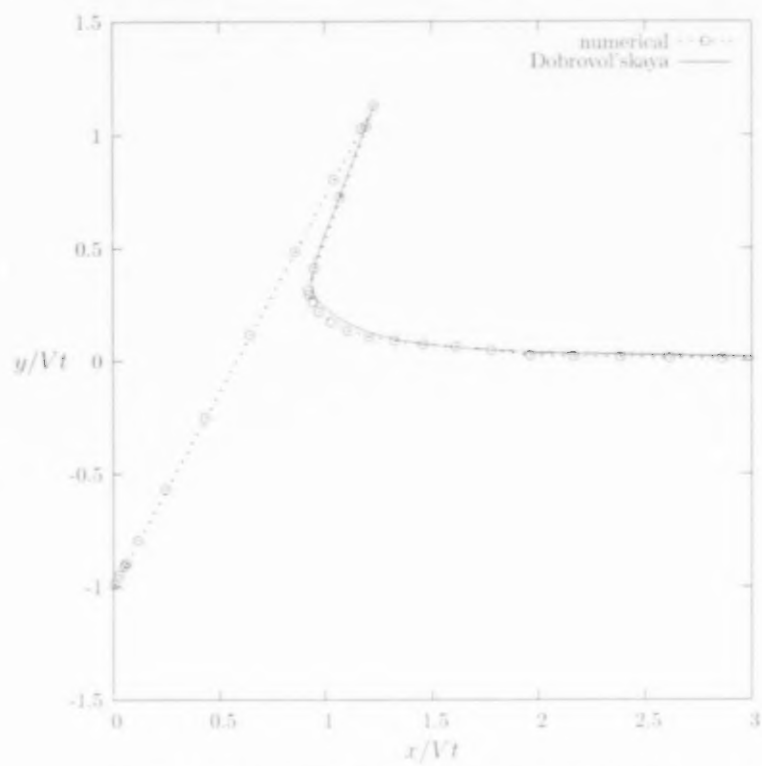


Figure 17: Comparison of self-similar flow for deadrise angle $6(i)''$

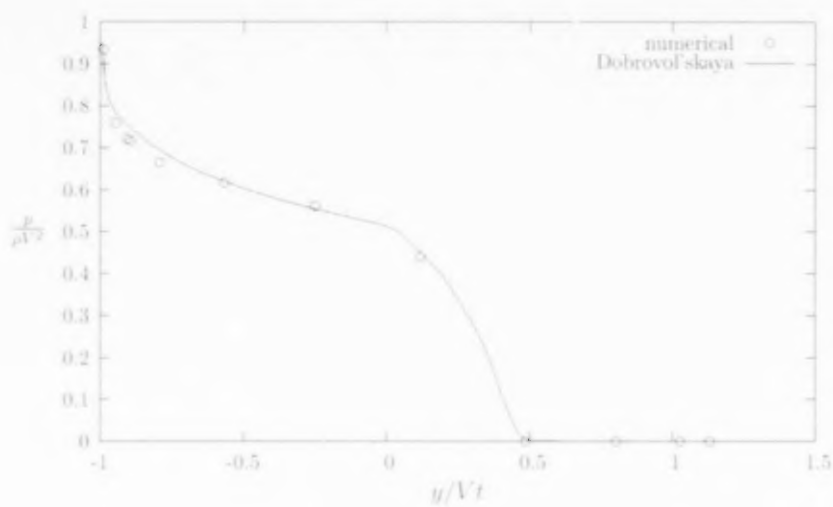


Figure 18: Comparison of pressure distribution in time independent domain for deadrise angle 60°

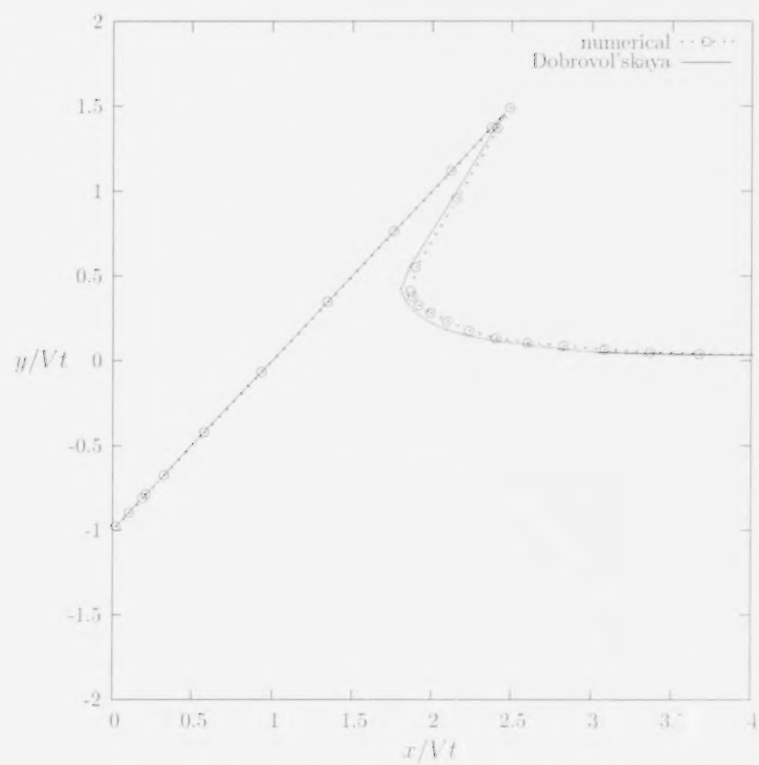


Figure 19: Comparison of self-similar flow for deadrise angle 45°

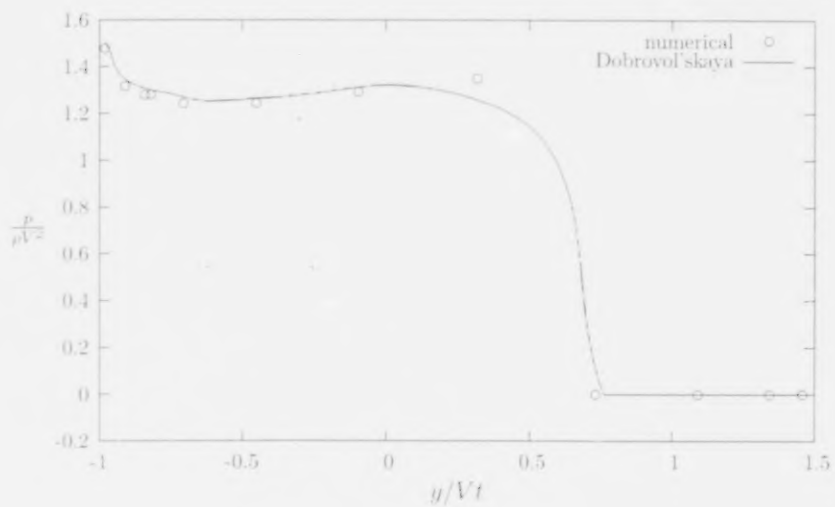


Figure 20: Comparison of pressure distribution in time independent domain for deadrise angle 45°

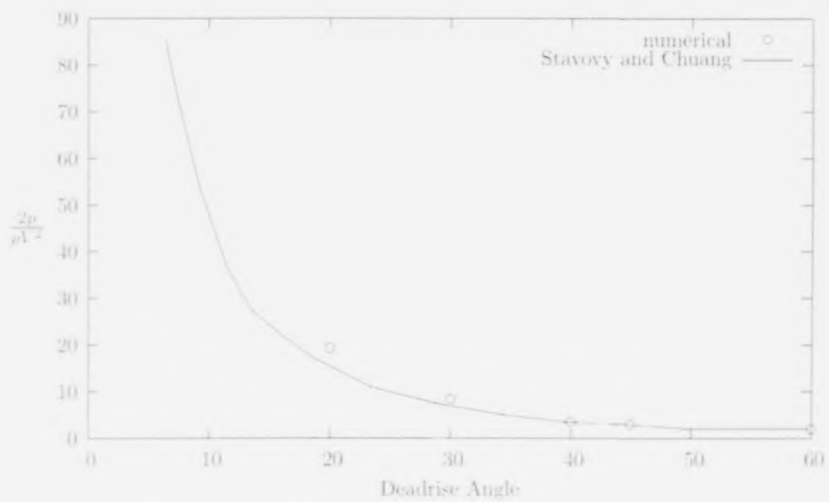


Figure 21: Slamming Pressure Coefficients for Wedge Sections

8 CPF Bow Flare Impulsive Load

After validation of our numerical scheme with wedge entry problem, this nonlinear numerical scheme based on desingularized Cauchy's formula demonstrates the robustness and accuracy. Since above scheme is only applicable for 2-D problems, strip theory is applied, so that several typical sections are taken from bow area to carry out the computation. Once the pressure distribution on each section is found out, the bow flare impulsive load can be determined.

As mentioned in chapter 2, the surface of CPF hull can be approximated by NURBS surface, therefore any curve of desired section can be interpolated easily from that NURBS surface. Three typical sections with bow flare are chosen and shown in following figures.

Three sections near the bow shown in Fig. 21 are selected for the computation. Their cross sectional curves are depicted in Fig. 22 :

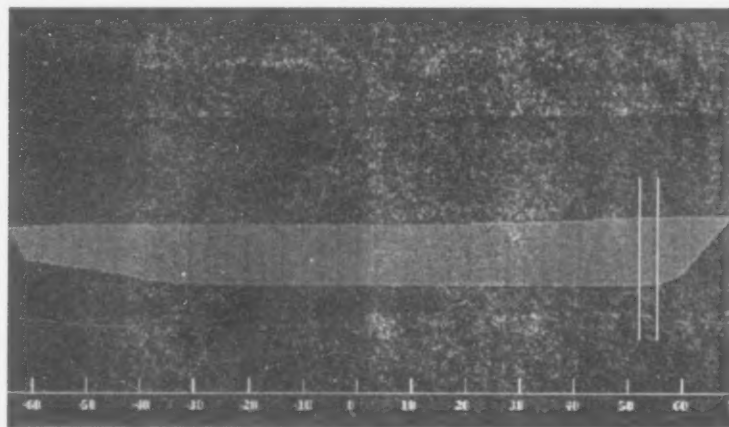


Figure 22: Cross Sectional Curve

Different from wedges the profiles of sections are curves, the numerical scheme has to be modified to be adaptable for their characteristics. As mentioned before, the section profiles obtained from NURBS surface obviously are NURBS curves. During the time marching those curves are functions of time t because the sections move down at a constant velocity V . Therefore the body surface (sections) can be written as following equations in NURBS form:

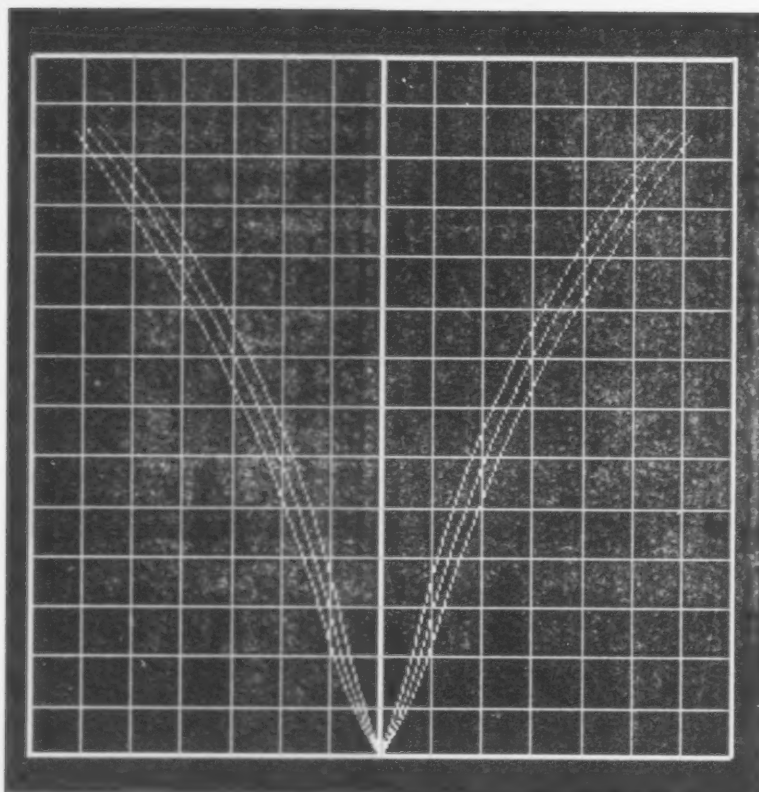


Figure 23: Cross Sectional Curve

$$x(\bar{u}) = \frac{\bar{N}_x(\bar{u})}{\bar{D}(\bar{u})} \quad (116)$$

$$y(\bar{u}) = \frac{\bar{N}_y(\bar{u})}{\bar{D}(\bar{u})} - Vt \quad (117)$$

where $\bar{N}_x(\bar{u}) = \sum_{i=0}^n N_i^{(p)}(\bar{u})w_iX_i$, $\bar{N}_y(\bar{u}) = \sum_{i=0}^n N_i^{(p)}(\bar{u})w_iY_i$ and $\bar{D}(\bar{u}) = \sum_{i=0}^n N_i^{(p)}(\bar{u})w_i$

Notice that X_i , Y_i are coordinates of the control point $\bar{P}_i$, above functions and variables are known from interpolation.

In every time step after solving the tangential velocities of fluid particles on body surface, only the particle at jet tip is traced. so that the predicted position of jet tip can be found for next time step, at same time the the total arc length of wetted portion of section curves and the parametric variable $\bar{u}$ at the intersection point between body surface and free surface can also be calculated by using the properties of NURBS curve. The properties of NURBS curve also can help to allocate the grid points along body curve with Gaussian quadrature and calculate the attributes of Gaussian points along the body curves for the following time step.

Fig 24., Fig 26. and Fig 28. show the development of jet flow on three selected typical sections with bow flare. Fig 25., Fig 27. and Fig 29. show the pressure distributions on wetted surface of above typical sections. From those figures one can find the impulsive force appears when the bow flare area enters water. That means the numerical results are consistent with the experimental observations in article of Zhao[10]. In this computation, all variables are nondimensionalized by the proper manners for computational convenience.

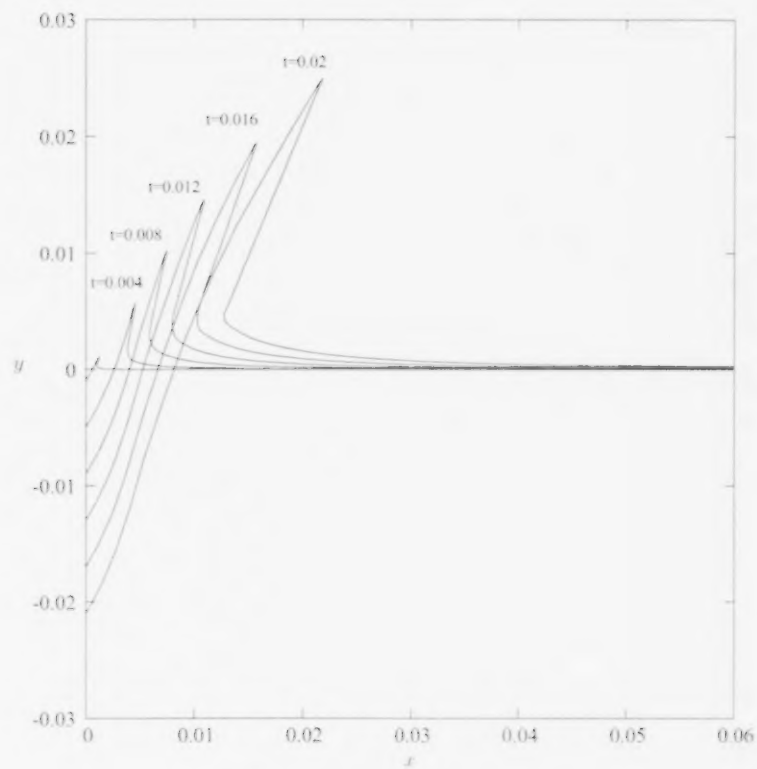


Figure 24: Development of jet flow on section 1

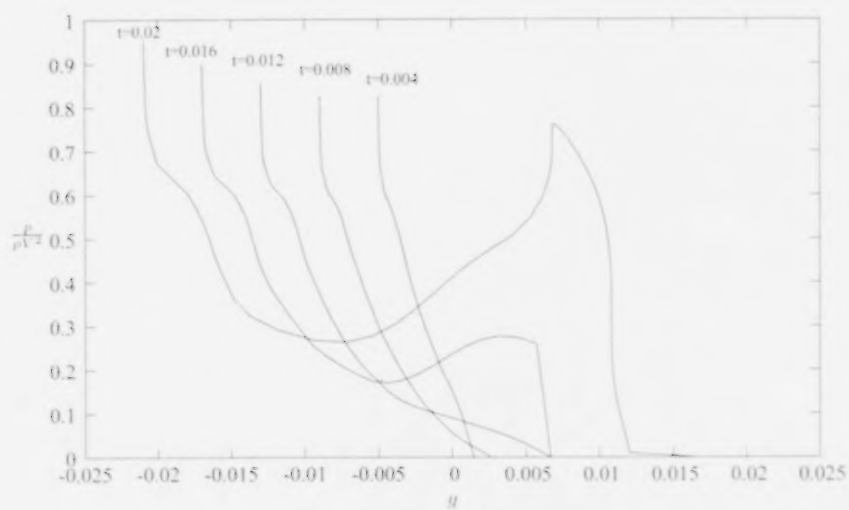


Figure 25: Development of pressure distribution on section 1

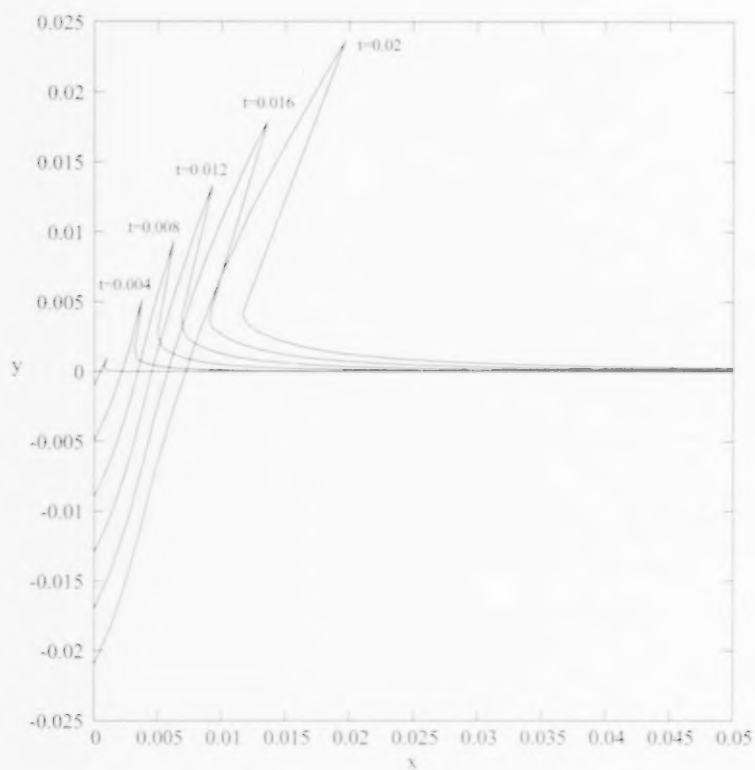


Figure 26: Development of jet flow on section2

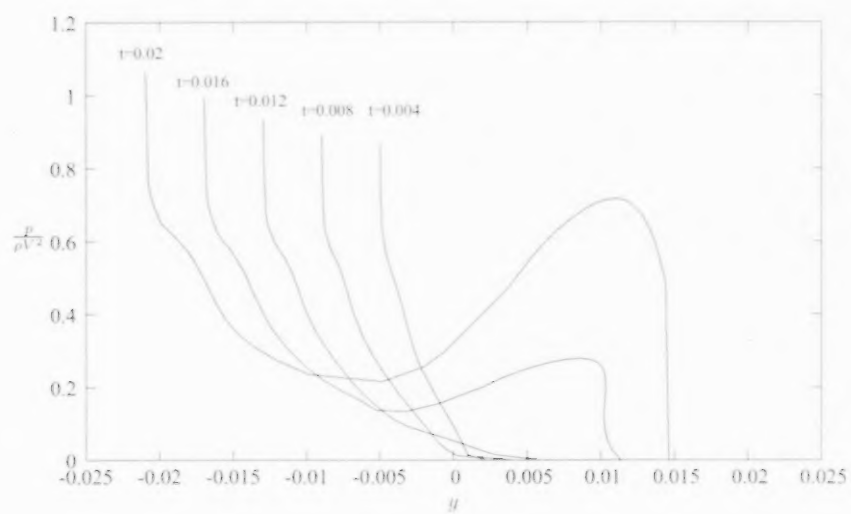


Figure 27: Development of pressure distribution on section 2

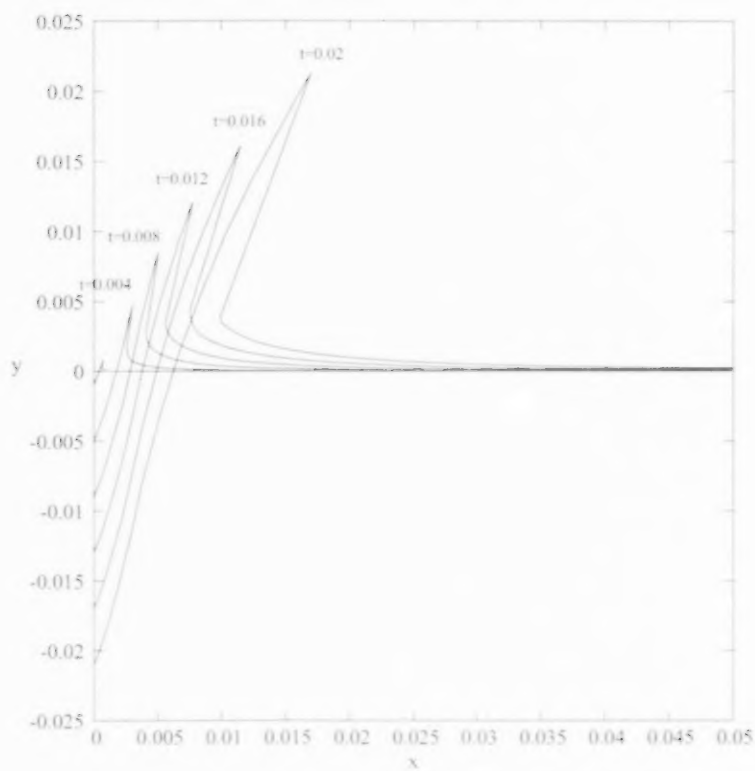


Figure 28: Development of jet flow on section3

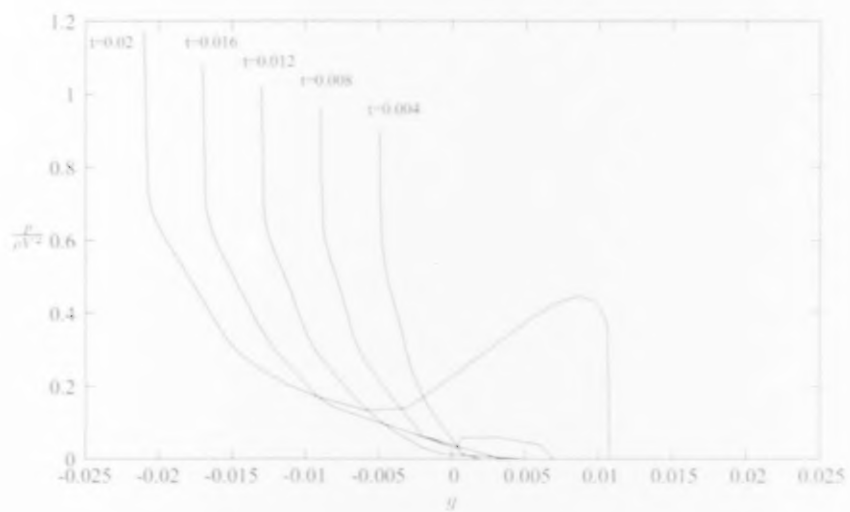


Figure 29: Development of pressure distribution on section 3

9 Conclusion and Remarks

Based on desingularized Cauchy's formula, Legendre and Labatto Gaussian quadratures, the 2-D nonlinear time domain free surface flow has been solved for the container and wedge entry problems. The treatment of the intersection points between different boundaries under different boundary condition has been dealt with in detail. Furthermore, the contribution to pressure distribution from the $\partial\phi/\partial t$ -term is obtained from solving another boundary value problem to improve the accuracy. The computation of CPF bow flare impulsive load on sections around CPF's bow were carried out. Furthermore, an interactive GUI [19] for computation, visualization and animation of computational results of water entry problem is also developed.

References:

1. J.M. Chuang and Jinhua Sheng, "Numerical Computation of Potential Flow Around Ship Hull Using NURBS and Regularized Boundary Integral Equation" (In preparation).
2. Ridley J.A. "A study of some theoretical aspects of slamming " *N.M.I. report No.R158* 1982.
3. Wagner H "Überstoss-und Gleitueergange ander Oberflache UonFlussigkeiter *Z.Angew Math Mech* 12(4) , pp. 192-235 1932.
4. Armand J.L. & Cointe R. "Hydrodynamic impact analysis of a cylinder. In Proc. Fifth Int Offshore Mech and Arctic Engng Symp" *Tokyo Japan Vol. 1*, pp. 609-634. *ASME* 1986.
5. Cointe.R "Free surface flows close to a surface -piercing body in mathematical approaches in hydrodynamics" pp. 319-334 *Soc. Ind. Appl. Maths Philadelphia, USA* 1991.
6. Howison S.D. Ockendon, J.R. & Wilson S.K. "Incompressible water entry problems at small deadrise angles" *J.Fluid Mech.* 222, pp. 215-230 1991.
7. Z.N. Dobrovol'skaya, "On Some Problems of Similarity Flow of Fluid with a Free Surface", *Journal of Fluid Mechanics*, Vol. 36, 1969, pp. 805-829.
8. Vinje,T & Brevig, P. "Nonlinear two-dimensional ship motions" *Proc. 3rd Int. Conf. Num. Ship Hydro. Paris June*, pp. 257-266, 198. 1980.
9. Greenhow,M. "Wedge entry into initially calm water" *Appl. Ocean Res.* 214-223 1987.
10. R.Zhao and O.Faltinsen "Water entry of two dimensional bodies" *J. Fluid Mech. Vol.246*, pp. 593-612 1993.
11. Lin, W-M., Newman, J.N. and Yue, D.K. "Nonlinear forced motions of floating bodies" *proc. 15th symp Naval Hydro. Hurnburg Germany Session1*, pp1-15 1984.
12. J.M. Chuang, "Application of desingularized Cauchy's formula to exterior potential problem using Gaussian quadrature". (In preparation)
13. S.M. Robinson and A.H. Stroud, "The approximate solution of an integral equation using hight-order Gaussian quadrature formula", *Math. Comput.* 15 (1961), 286-288

14. L. Piegl and W. Tiller, "The NURBS Book", Second Edition, Springer, New York, 1997.
15. J. Neider, T. Davis, M. Woo, "OpenGL Programming Guide", Release 1, Addison-Wesley Publishing Company, New York, 1993.
16. M.K. Dalheimer, "Programming with Qt", 2nd Edition, O'Reilly & Associates, 2002.
17. G.X. Wu and R.Eatock Taylor, "Finite element analysis of two-dimensional non-linear transient water waves", *Applied Ocean Research*, 16, 1994, pp. 363-372.
18. Kevin McTaggart, "SHIPMO7: An Updated Strip Theory Program for Predicting Ship Motions and Sea Loads in Waves", DREA Technical Memorandum 96/243 March 1997.
19. J.M. Chuang and Qin Tu, "User's Manual of Interactive Software Package for Predicting CPF Bow-Flare Impulsive Loads", Technical Report, Dalhousie University, Feb, 2004.

Annex A

Appendix A

The analytical solution of the free surface elevation $f(x) = \eta(x)$ for similarity flow for wedges derived in Section 3 is expressed in terms of a nonlinear integral equation:

$$f(t) = \frac{1}{\pi} \frac{e_0^2}{e^2} \int_0^t \frac{(1-\bar{t})^{1-\gamma} \exp \left\{ \bar{t} \int_0^1 \frac{f(\tau)}{\tau(\tau-\bar{t})} d\tau \right\}}{\int_{\bar{t}}^1 s^{-\frac{3}{2}} (1-s)^{-\frac{1}{2}+\gamma} \exp \left\{ -s \int_0^1 \frac{f(\tau)}{\tau(\tau-s)} d\tau \right\} ds} d\bar{t} \quad (\text{A.1})$$

A.1 Investigation of Numerator of Singular Integral Equation

$$\begin{aligned} I_N(t) &= (1-t)^{-1-\gamma} \exp \left\{ t \int_0^1 \frac{f(\tau)}{\tau(\tau-t)} d\tau \right\} \\ &= (1-t)^{-1-\gamma} \exp \left\{ t \left[\frac{1}{t} \left(\int_0^1 \frac{f(\tau)}{\tau-t} d\tau - \int_0^1 \frac{f(\tau)}{\tau} d\tau \right) \right] \right\} \\ &= (1-t)^{-1-\gamma} \exp \left\{ \int_0^1 \frac{f(\tau)}{\tau-t} d\tau - \int_0^1 \frac{f(\tau)}{\tau} d\tau \right\} \end{aligned} \quad (\text{A.2})$$

where $\gamma = \frac{1}{2} - \frac{\alpha}{\pi}$

The first integral can be written in a desingularized form

$$\begin{aligned} I_1(t) &= \int_0^1 \frac{f(\tau)}{\tau-t} d\tau \\ &= \int_0^1 \frac{f(\tau) - f(t)}{\tau-t} d\tau + f(t) \int_0^1 \frac{1}{\tau-t} d\tau \\ &= \int_0^1 \frac{f(\tau) - f(t)}{\tau-t} d\tau + f(t) \log \frac{1-t}{t} \end{aligned} \quad (\text{A.3})$$

This equation is valid for $0 \leq t < 1$.

The second integral can be rewritten as

$$I_2(t) = \int_0^1 \frac{f(\tau)}{\tau} d\tau = \int_0^1 \frac{f(\tau) - f(0)}{\tau-0} d\tau + f(0) \log \frac{1}{0} \quad (\text{A.4})$$

Since

$$\lim_{t \rightarrow 0} f(t) = O(t^{3/2})$$

$I_2(t)$ is reduced to

$$I_2(t) = \int_0^1 \frac{f(\tau) - f(0)}{\tau - 0} d\tau \quad (\text{A.5})$$

When $t \rightarrow 0$, the integrand of $I(t)$ becomes

$$\lim_{t \rightarrow 0} \left\{ \frac{f(\tau)}{\tau - t} - \frac{f(\tau)}{\tau} \right\} = 0 \quad (\text{A.6})$$

Finally, the integral I can be written in a desingularized form as below

$$\begin{aligned} I_N(t) &= (1-t)^{-1-\gamma} \\ &\times \exp \left\{ \int_0^1 \frac{f(\tau) - f(t)}{\tau - t} d\tau - \int_0^1 \frac{f(\tau) - f(0)}{\tau - 0} d\tau + f(t) \log \frac{1-t}{t} \right\} \\ &= (1-t)^{-1-\gamma} \left[\frac{1-t}{t} \right]^{f(t)} \\ &\times \exp \left\{ \int_0^1 \frac{f(\tau) - f(t)}{\tau - t} d\tau - \int_0^1 \frac{f(\tau) - f(0)}{\tau - 0} d\tau \right\} \\ &= (1-t)^{-1-\gamma+f(t)} t^{-f(t)} \\ &\times \exp \left\{ \int_0^1 \frac{f(\tau) - f(t)}{\tau - t} d\tau - \int_0^1 \frac{f(\tau) - f(0)}{\tau - 0} d\tau \right\} \quad (\text{A.7}) \end{aligned}$$

A few observations are summarized as follows:

1. When $t \rightarrow 0$

$$I_N(0) = \lim_{t \rightarrow 0} (1-t)^{-1-\gamma} \exp \left\{ \int_0^1 \frac{f(\tau)}{\tau - t} d\tau - \int_0^1 \frac{f(\tau)}{\tau} d\tau \right\} = 1 \quad (\text{A.8})$$

2. When $t \rightarrow 1$

$$\begin{aligned} I_N(1) &= \lim_{t \rightarrow 1} (1-t)^{-1-\gamma+f(t)} t^{-f(t)} \\ &\times \exp \left\{ \int_0^1 \frac{f(\tau) - f(t)}{\tau - t} d\tau - \int_0^1 \frac{f(\tau) - f(0)}{\tau - 0} d\tau \right\} \\ &= \lim_{t \rightarrow 1} \left\{ (1-t)^{-1-\gamma+f(t)} \right\} \\ &\times \exp \left\{ \int_0^1 \frac{f(\tau) - f(1)}{\tau - 1} d\tau - \int_0^1 \frac{f(\tau) - f(0)}{\tau - 0} d\tau \right\} \quad (\text{A.9}) \end{aligned}$$

From Eqn. (5.3) of Ref. [7], it shows that

$$\frac{1}{4} + \gamma < f(1) < \frac{1}{2} + \gamma$$

Thus, we have

$$-\frac{3}{4} \leq -1 - \gamma + f(1) < -\frac{1}{2}$$

A.2 Investigation of Denominator of Singular Integral Equation

The denominator of the nonlinear integral equation is given by

$$I = \int_t^1 s^{-3/2}(1-s)^{-a} \exp \left\{ -s \int_0^1 \frac{f(\tau)}{\tau(\tau-s)} d\tau \right\} ds \quad (\text{A.10})$$

where $a = \alpha/\pi$ and $0 \leq \alpha \leq \pi/2$, thus $0 \leq a \leq 1/2$

Let

$$F(s) = \exp \left\{ -s \int_0^1 \frac{f(\tau)}{\tau(\tau-s)} d\tau \right\} \quad (\text{A.11})$$

Then, Eqn (A.10) is written as

$$I = \int_t^1 s^{-3/2}(1-s)^{-a} F(s) ds \quad (\text{A.12})$$

It is obvious that when $s = 1$ there is a singularity in the integrand. To remove this singularity, we define

$$\begin{aligned} Z(s) &= s^{-3/2}(1-s)^{-a} F(s) \\ \phi(s) &= s^{-3/2} F(s) \\ X(s) &= \phi(1)(1-s)^{-a} + \phi'(1)(1-s)^{1-a} \end{aligned} \quad (\text{A.13})$$

Eqn. (A.12) can be rewritten as

$$I = \int_t^1 \left\{ Z(s) - X(s) \right\} ds + \int_t^1 X(s) ds \quad (\text{A.14})$$

It can be seen that

$$\begin{aligned} \lim_{s \rightarrow 1} \left\{ Z(s) - X(s) \right\} &= \\ \lim_{s \rightarrow 1} \left\{ s^{-3/2}(1-s)^{-a}F(s) - \left[\phi(1)(1-s)^{-a} + \phi'(1)(1-s)^{1-a} \right] \right\} \\ &= 0 \end{aligned} \quad (\text{A.15})$$

Notice that

$$\phi(1) = F(1), \quad \phi'(1) = -\frac{3}{2}F(1) + F'(1)$$

Therefore, in Eqn. (A.14) the first integral is singularity free and the second integral can be carried out analytically for $t > 0$,

$$\begin{aligned} \int_t^1 X(s)ds &= \phi(1) \int_t^1 (1-s)^{-a}ds + \phi'(1) \int_t^1 (1-s)^{1-a}ds \\ &= -\frac{\phi(1)}{1-a}(1-s)^{1-a} \Big|_t^1 - \frac{\phi'(1)}{2-a}(1-s)^{2-a} \Big|_t^1 \\ &= \frac{\phi(1)}{1-a}(1-t)^{1-a} + \frac{\phi'(1)}{2-a}(1-t)^{2-a} \end{aligned} \quad (\text{A.16})$$

$$\begin{aligned} I_D(t) &= t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}+\gamma} \exp \left\{ -t \int_0^1 \frac{f(\tau)}{\tau(\tau-t)} d\tau \right\} \\ &= t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}+\gamma} \exp \left\{ -t \left[\frac{1}{t} \left(\int_0^1 \frac{f(\tau)}{\tau-t} d\tau - \int_0^1 \frac{f(\tau)}{\tau} d\tau \right) \right] \right\} \\ &= t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}+\gamma} \exp \left\{ - \left[\int_0^1 \frac{f(\tau)}{\tau-t} d\tau - \int_0^1 \frac{f(\tau)}{\tau} d\tau \right] \right\} \end{aligned} \quad (\text{A.17})$$

where $\gamma = \frac{1}{2} - \frac{a}{2}$

Based on Eqn. (A.7), we can write

$$\begin{aligned} I_D(t) &= t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}+\gamma} \\ &\quad \times \exp \left\{ - \left[\int_0^1 \frac{f(\tau)-f(t)}{\tau-t} d\tau - \int_0^1 \frac{f(\tau)-f(0)}{\tau-0} d\tau + f(t) \log \frac{1-t}{t} \right] \right\} \end{aligned}$$

$$\begin{aligned}
&= t^{-\frac{3}{2}}(1-t)^{-\frac{1}{2}+\gamma} \left[\frac{1-t}{t} \right]^{-f(t)} \\
&\quad \times \exp \left\{ - \left[\int_0^1 \frac{f(\tau) - f(t)}{\tau - t} d\tau - \int_0^1 \frac{f(\tau) - f(0)}{\tau - 0} d\tau \right] \right\} \\
&= t^{-\frac{3}{2}+f(t)}(1-t)^{-\frac{1}{2}+\gamma-f(t)} \\
&\quad \times \exp \left\{ - \left[\int_0^1 \frac{f(\tau) - f(t)}{\tau - t} d\tau - \int_0^1 \frac{f(\tau) - f(0)}{\tau - 0} d\tau \right] \right\} \quad (\text{A.18})
\end{aligned}$$

A.3 Discretization

Choosing Gaussian point as the collocation point, the integral I can be carried out as

$$I_i = \sum_{j=0, j \neq i}^N \left\{ \frac{f_j - f_i}{\tau_j - \tau_i} \right\} w_j + f_i' w_i - \sum_{j=1}^N \frac{f_j - f_0}{\tau_j - \tau_0} w_j - f_0' w_0 + f_i \log \frac{1 - \tau_i}{\tau_i} \quad (\text{A.19})$$

Note:

1. For numerical integration, the quadrature should include two end point, such as Chebychev quadrature.
2. The collocation point in t just can be distributed from 0 to a very close point to 1, namely, $1 - \epsilon$, where ϵ is a very small number. The is due to the fact that the log term in I_1 approaches to ∞ when $t = 1$.

Annex B

Appendix B Analytical Solution of Container Problem

The following analytical solution can be found in [17]:

$$\phi_1 = -\frac{ag \cosh k_2(y+d)}{\omega_2 \cosh k_2 d} \cos k_2 x \sin \omega_2 t \quad (\text{B.1})$$

$$\psi_1 = \frac{ag \cosh k_2(y+d)}{\omega_2 \sinh k_2 d} \sin k_2 x \sin \omega_2 t \quad (\text{B.2})$$

$$\eta_1 = a \cos \omega_2 t \cos k_2 x \quad (\text{B.3})$$

$$\frac{\partial \phi_1}{\partial t} = -ag \frac{\cosh k_2(y+d)}{\cosh k_2 d} \cos k_2 x \cos \omega_2 t \quad (\text{B.4})$$

$$\frac{\partial \psi_1}{\partial t} = ag \frac{\sinh k_2(y+d)}{\cosh k_2 d} \sin k_2 x \cos \omega_2 t \quad (\text{B.5})$$

$$\eta_2 = -\frac{1}{g} \left\{ \frac{\partial \phi_2}{\partial t} + \eta_1 \frac{\partial \phi_1}{\partial y \partial t} + \frac{1}{2} \nabla \phi_1 \cdot \nabla \phi_1 \right\} \quad (\text{B.6})$$

$$\begin{aligned} \phi_2 = & \frac{1}{16} \frac{a^2}{\omega_2^2} (3\omega_2^4 + k_2^2 g^2) \sin 2\omega_2 t + \\ & \frac{1}{16} \frac{2^2}{\omega_2^3} (\omega_2^4 - k_2^2 g^2)^2 \frac{\cosh 2k_2(y+d)}{\cosh k_2 d} \cos k_2 x \sin 2\omega_2 t + D_0 t + \\ & D_4 \frac{\cosh k_4(y+d)}{\cosh k_4 d} \cos k_4 x \sin \omega_4 t \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} \frac{\partial \phi_2}{\partial t} = & 2\omega_2 \left\{ \frac{1}{16} \frac{a^2}{\omega_2^2} (3\omega_2^4 + k_2^2 g^2) \cos 2\omega_2 t + \right. \\ & \left. \frac{1}{16} \frac{2^2}{\omega_2^3} (\omega_2^4 - k_2^2 g^2)^2 \frac{\cosh 2k_2(y+d)}{\cosh k_2 d} \cos k_2 x \cos 2\omega_2 t \right\} + D_0 + \\ & \omega_4 D_4 \frac{\cosh k_4(y+d)}{\cosh k_4 d} \cos k_4 x \cos \omega_4 t \end{aligned} \quad (\text{B.8})$$

$$\frac{\partial^2 \phi_1}{\partial t \partial y} = -agk_2 \frac{\sinh k_2(y+d)}{\cosh k_2 d} \cos k_2 x \cos \omega_2 t \quad (\text{B.9})$$

$$\nabla \phi_1 = \left\{ \frac{agk_2 \cosh k_2(y+d)}{\omega_2 \cosh k_2 d} \sin k_2 x \sin \omega_2 t \right\} \vec{i} -$$

$$\left\{ \frac{agk_2 \sinh k_2(y+d)}{\omega_2 \cosh k_2 d} \cos k_2 x \sin \omega_2 t \right\}_j \quad (\text{B.10})$$

Where ϕ_1 is the linear analytical solution of velocity potential. By using Stokes perturbation expansion of the velocity potential, ϕ_1 can be regarded as the first order term, ϕ_2 is the second order term and can be applied to describe the nonlinear phenomena.

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In this research, the theoretical background and the numerical scheme for predicting CPF bow-flare impulsive loads are studied and developed, respectively. The geometry of CPF is first approximated by NURBS surface. It is mainly used to find any CPF's cross section around the bow for the computation of 2-D impulsive load distribution. To validate the developed numerical scheme for solving nonlinear free surface problem in the time-domain which is associated with the prediction of CPF's impulsive load, the similarity transformation of wedge entry problem, a container problem and the Euler-Lagrangian method are derived and investigated in detail. From numerical experiments, the numerical solutions of these problems are in good agreement with the analytical solution.

Furthermore, using Qt (GUI-toolkits) and OpenGL (3-D rendering library), an interactive software, namely, CPFSLAM, is also developed for solving and visualizing wedge water entry, container problems as well as predicting CPF bow-flare impulsive load.

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bow-flare
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slamming

